Products and Ratios of Two Gaussian Class Correlated Weibull Random Variables

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Abstract. Starting from on a recently introduced Gaussian class bivariate Weibull stochastic model, the probability density and the cumulative distribution functions of the product $(Z_1 Z_2)^c$ and the ratio $(Z_1/Z_2)^c$, when Z_1 and Z_2 are correlated Weibull random variables belonging to this class ($c > 0$), are derived in closed form. Moreover, using the inequality between arithmetic and geometric mean, a union upper bound for the distribution of the sum of two correlated Weibull variates $Z_1^c + Z_2^c$ is also presented. Special cases of our results are in agreement with previously published ones. The proposed analysis is useful in several scientific fields of engineering.

Keywords: Correlated statistics, distribution of product, distribution of ratio, stochastic models, Weibull.

1 Introduction

The Weibull distribution was first introduced by Waloddi Weibull back in 1937 for estimating machinery lifetime and became widely known in 1951 [Weibull, 1951]. Nowadays, the Weibull distribution is used in several fields of science. For example, it is a very popular statistical model in reliability engineering and failure data analysis. It is also used in many other fields of science, such as weather forecasting and data fitting of all kinds, while it is widely applied in radar systems to model the dispersion of the received signals level produced by some types of clutters [Sekine and Mao, 1990]. Interestingly enough, the Weibull distribution has become popular in the scientific field of communications engineering, since it seems to exhibit good fit to experimental channel measurements (see [Sagias and Karagiannidis, 2005] and references therein).

Over the past, there have been published many papers in the open technical literature, where the distributions of products and ratios are studied for independent [Nadarajah and Kotz, 2006], [Steece, 1976], [Nadarajah, 2005], [Glena *et al.*, 2004], [Nadarajah and Kotz, 2005], [Nadarajah and Ali, 2005], [Nadarajah and Gupta, 2005], [Mathai, 1972] and correlated [Simon, 2002], [Nakagami and Ōta, 1957], [Malik and Trudel, 1986], [Nakagami, 1960] random variables (RV)s. Such products and ratios are very useful to biological and physical sciences, econometrics, classification, ranking, and selection. Also, research efforts have been made to derive several classes of bivariate Weibull distributions (see [Kotz *et al.*, 2000, Chapter 47/4] and references therein). For example, two very popular bivariate Weibull models are the Gumbel's [Gumbel, 1960] and Freund's [Freund, 1961] bivariate exponentials. Recently, a new class of Weibull distributions has been introduced [Sagias and Karagiannidis, 2005], where the RVs of that class are originated from correlated Gaussian processes. To the best of the authors' knowledge, an analysis for the distributions of products or ratios considering that novel model has not been published yet.

In this paper, based on the bivariate Weibull distribution of the Gaussian class presented in [Sagias and Karagiannidis, 2005], we obtain the distributions of the product and the ratio of two correlated Weibull RVs. More specifically, their probability density functions (PDF)s and cumulative distribution functions (CDF)s are extracted in closed form. Furthermore, a useful union upper bound for the distribution of the sum of two correlated Weibull variates is also presented.

The remainder of this paper is organized as following: In Sections 2 and 3, the distributions of the products and ratios of two correlated RVs are presented, respectively. In Section 4, a useful bound for the distribution of the sum of two correlated Weibull RVs is obtained, while concluding remarks are provided in Section 5.

2 Distribution of the product of two Weibull rvs

Let $Z_\ell \geq 0$ ($\ell = 1$ and 2) be two, not necessarily identically distributed, Weibull RVs having joint PDF

$$\begin{aligned} f_{Z_1,Z_2}(y_1,y_2) = & \frac{\beta^2\,(y_1\,y_2)^{\beta-1}}{\Omega_1\,\Omega_2\,(1-\rho)} \\ & \times \exp\left[-\frac{1}{1-\rho}\left(\frac{y_1^\beta}{\Omega_1}+\frac{y_2^\beta}{\Omega_2}\right)\right] I_0\left[\frac{2\sqrt{\rho}\,(y_1\,y_2)^{\beta/2}}{(1-\rho)\sqrt{\Omega_1\,\Omega_2}}\right] \end{aligned} \tag{1}$$

with $\beta > 0$ and $\Omega_\ell = \mathbb{E}\langle Z_\ell^\beta\rangle > 0$ ($\mathbb{E}\langle\cdot\rangle$ denoting expectation) being the shaping and scaling parameters, respectively, and $I_n(\cdot)$ being the nth ($n \in \mathbb{N}$) order modified Bessel of the first kind [Gradshteyn and Ryzhik, 2000, eq. (8.406)]. For $\beta = 2$ and $\beta = 1$, (1) includes the well-known Rayleigh and negative exponential PDFs, respectively, as special cases. Also, the Weibull power correlation coefficient, i.e., between Z_1^2 and Z_2^2, is related with the Rayleigh power correlation coefficient ($0 \leq \rho < 1$) as

$$\varrho = \frac{{}_2F_1(-1/\beta, -1/\beta; 1; \rho) - 1}{-1 + \Gamma(1 + 2/\beta)/\Gamma^2(1 + 1/\beta)} \tag{2}$$

with $0 \leq \varrho < 1$ and ${}_pF_q(\cdot)$ representing the generalized hypergeometric function with p, q integers [Gradshteyn and Ryzhik, 2000, eq. (9.14/1)]. The bivariate model described by (1) assumes that Z_1 and Z_2 have two parameters Weibull marginal PDFs given by[1]

$$f_{Z_\ell}(y) = \frac{\beta}{\Omega_\ell}\, y^{\beta-1} \exp\left(-\frac{y^\beta}{\Omega_\ell}\right). \tag{3}$$

We define a RV $\mathcal{P} \triangleq (Z_1\, Z_2)^c$ with $c > 0$ being a positive constant value.

Theorem 1 (Probability density function). *The PDF of $\mathcal{P}$ is given by*

$$f_{\mathcal{P}}(y) = \frac{2\,\beta\, y^{\beta/c-1}}{c\,\lambda_1\,(1-\rho)}\, I_0\left[\frac{2\sqrt{\rho}\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_0\left[\frac{2\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] \tag{4}$$

with $\lambda_1 = \Omega_1\,\Omega_2$ and $K_n(\cdot)$ be the nth order modified Bessel of the second kind [Gradshteyn and Ryzhik, 2000, eq. (8.407)].

Proof. By using [Papoulis, 2001, eq. (6-74)] and (1), and after simple algebraic manipulations yields (4).

By letting $c = 1$, for $\rho = 0$, the first Bessel function equals to one ($I_0(0) = 1$), and hence, (4) reduces to $f_{\mathcal{P}}(y) = 2\,\lambda_1^{-1}\,\beta\, y^{\beta-1}\, K_0\left(2\, y^{\beta/2}/\sqrt{\lambda_1}\right)$, which agrees with a known result [Nakagami, 1960, eq. (90)], while for $\Omega_1 = \Omega_2 = 1$ ($\lambda_1 = 1$), also agrees[2] with [Mathai, 1972, eq. (3.2)] concerning the product of independent and identical Weibull RVs. Furthermore for $\beta = 2$, (4) simplifies to [Nakagami, 1960, eq. (145)], [Simon, 2002, eq. (6.55)]

$$f_{\mathcal{P}}(y) = \frac{4\, y}{\lambda_1\,(1-\rho)}\, I_0\left[\frac{2\sqrt{\rho}\, y}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_0\left[\frac{2\, y}{\sqrt{\lambda_1}\,(1-\rho)}\right]. \tag{5}$$

Lemma 1 (Cumulative distribution function). *The CDF of $\mathcal{P}$ is given by*

$$\begin{aligned} F_{\mathcal{P}}(y) = 1 - \frac{2\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)} \Bigg\{ & I_0\left[\frac{2\sqrt{\rho}\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_1\left[\frac{2\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] \\ & + \sqrt{\rho}\, I_1\left[\frac{2\sqrt{\rho}\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_0\left[\frac{2\, y^{\beta/(2c)}}{\sqrt{\lambda_1}\,(1-\rho)}\right] \Bigg\}. \end{aligned} \tag{6}$$

[1] A Weibull RV Z_ℓ with PDF given by (3), is denoted as $\mathcal{W}_{Z_\ell}(\beta, \Omega_\ell)$.

[2] A term is missing in [Mathai, 1972, eq. (4.6)].

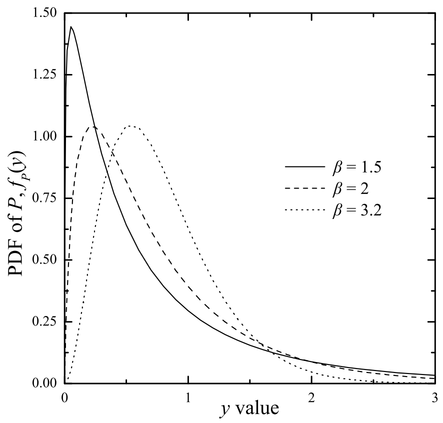

Fig. 1. PDF of the product of two correlated RVs for $\beta = 1.5$, 2, and 3.2.

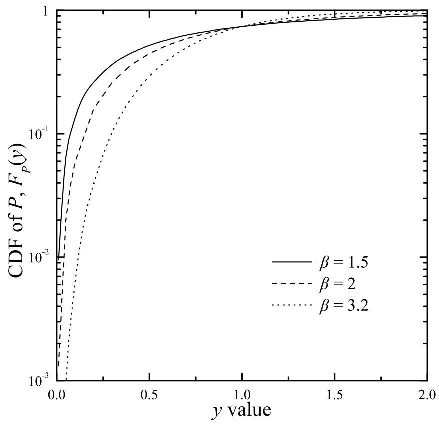

Fig. 2. CDF of the product of two correlated RVs for $\beta = 1.5$, 2, and 3.2.

Proof. By integrating $f_{\mathcal{P}}(y)$ given by (4) from zero to y, applying the transformation $w = y^{\beta/c}$, and using [Gradshteyn and Ryzhik, 2000, eq. (5.54/1)], yields (6).

Corresponding special cases such as those for the PDF are the following ($c = 1$): For $\rho = 0$, (6) reduces to $F_{\mathcal{P}}(y) = 1 - 2\,\lambda_1^{-1/2}\,y^{\beta/2}\,K_1\left(2\,\lambda_1^{-1/2}\,y^{\beta/2}\right)$, while for $\beta = 2$ simplifies to

$$F_{\mathcal{P}}(y) = 1 - \frac{2\,y}{\sqrt{\lambda_1}\,(1-\rho)} \left\{ I_0\left[\frac{2\sqrt{\rho}\,y}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_1\left[\frac{2\,y}{\sqrt{\lambda_1}\,(1-\rho)}\right] + \sqrt{\rho}\,I_1\left[\frac{2\sqrt{\rho}\,y}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_0\left[\frac{2\,y}{\sqrt{\lambda_1}\,(1-\rho)}\right]\right\}. \tag{7}$$

Without loss of the generality let $c = 1$. By numerically evaluating (4) and (6), the PDF and the CDF of $\mathcal{P}$ are presented in Figs. 1 and 2, respectively, as a function of y for $\Omega_1 = 1.2$, $\Omega_2 = 0.7$ ($\lambda_1 = 0.84$), $\rho = 0.25$, and several values of β.

3 Distribution of the ratio of two Weibull rvs

We define another RV as $\mathcal{R} \triangleq (Z_1/Z_2)^c$.

Theorem 2 (Probability density function). *The PDF of* $\mathcal{R}$ *is given by*

$$f_{\mathcal{R}}(y) = \frac{\beta\,\lambda_2\,(1-\rho)\,\left(1+\lambda_2\,y^{\beta/c}\right)\,y^{\beta/c-1}}{c\,\left[1 - 2\,\lambda_2\,(2\,\rho-1)\,y^{\beta/c} + \lambda_2^2\,y^{2\beta/c}\right]^{3/2}} \tag{8}$$

with $\lambda_2 = \Omega_2/\Omega_1 > 0$.

Proof. By using [Papoulis, 2001, eq. (6-43)] and (1), applying a transformation $w = y^{\beta/c}$, and after simple algebraic manipulations yields (8).

Letting $c = 1$, for $\rho = 0$, the above PDF reduces to the formula below [Mathai, 1972, eq. (4.10)] $f_{\mathcal{R}}(y) = \beta\,\lambda_2\,y^{\beta-1}/\left(1 + \lambda_2\,y^{\beta}\right)^2$ concerning the ratio of two independent Weibull RVs, which also agrees with [Nakagami, 1960, eq. (92)]. Also for $\beta = 2$, (4) simplifies to [Nakagami, 1960, eq. (145)]

$$f_{\mathcal{R}}(y) = \frac{2\,\lambda_2\,(1-\rho)\,\left(1 + \lambda_2\,y^2\right)\,y}{\left[1 - 2\,\lambda_2\,(2\,\rho - 1)\,y^2 + \lambda_2^2\,y^4\right]^{3/2}} \tag{9}$$

for the ratio of two correlated Rayleigh distributed RVs.

Lemma 2 (Cumulative distribution function). *The CDF of* $\mathcal{R}$ *is given by*

$$F_{\mathcal{R}}(y) = \frac{1}{2}\left[1 - \frac{1 - \lambda_2\,y^{\beta/c}}{\sqrt{1 - 2\,\lambda_2\,(2\,\rho - 1)\,y^{\beta/c} + \lambda_2^2\,y^{2\beta/c}}}\right]. \tag{10}$$

Proof. By integrating $f_{\mathcal{R}}(y)$ in (8) from zero to y, applying the transformation $w = y^{\beta/c}$, and using [Gradshteyn and Ryzhik, 2000, eqs. (2.264/5) and (2.264/5)], yields (10).

Corresponding special cases such as those for the PDF are the following ($c = 1$): For $\rho = 0$, (10) reduces to $F_{\mathcal{R}}(y) = \lambda_2\,y^{\beta}/\left(1 + \lambda_2\,y^{\beta}\right)$, while for $\beta = 2$ simplifies to

$$F_{\mathcal{R}}(y) = \frac{1}{2}\left[1 - \frac{1 - \lambda_2\,y^2}{\sqrt{1 - 2\,\lambda_2\,(2\,\rho - 1)\,y^2 + \lambda_2^2\,y^4}}\right]. \tag{11}$$

Having numerically evaluated (8) and (10) ($c = 1$), a few curves for the PDF and the CDF of $\mathcal{R}$ are presented in Figs. 3 and 4, respectively, as a function of y considering the same set of parameters as in Figs. 1 and 2 ($\lambda_2 = 0.583$).

4 An upper bound for the distribution of the sum of correlated Weibull rvs

Let us define a RV as $S \triangleq Z_1^c + Z_2^c$.

Theorem 3 (A CDF bound). *An upper bound for the CDF of* S *is*

$$F_S(y) \leq 1 - \frac{2\,(y/2)^{\beta/c}}{\sqrt{\lambda_1}\,(1-\rho)}\left\{I_0\left[\frac{2\,\sqrt{\rho}\,(y/2)^{\beta/c}}{\sqrt{\lambda_1}\,(1-\rho)}\right]\,K_1\left[\frac{2\,(y/2)^{\beta/c}}{\sqrt{\lambda_1}\,(1-\rho)}\right] + \sqrt{\rho}\,I_1\left[\frac{2\,\sqrt{\rho}\,(y/2)^{\beta/c}}{\sqrt{\lambda_1}\,(1-\rho)}\right]\,K_0\left[\frac{2\,(y/2)^{\beta/c}}{\sqrt{\lambda_1}\,(1-\rho)}\right]\right\}. \tag{12}$$

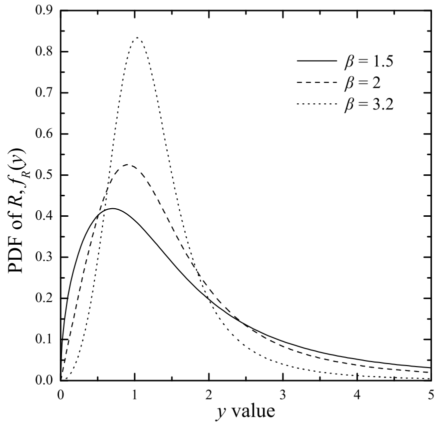

Fig. 3. PDF of the ratio of two correlated RVs for $\beta = 1.5$, 2, and 3.2.

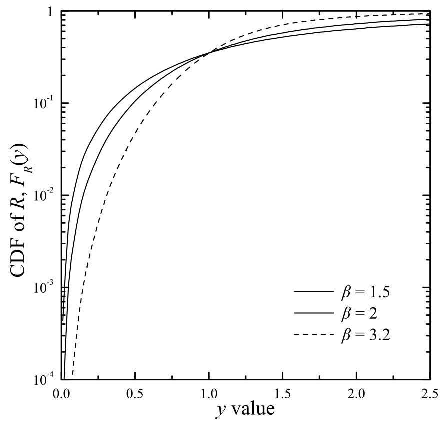

Fig. 4. CDF of the ratio of two correlated RVs for $\beta = 1.5$, 2, and 3.2.

Proof. Using the well-known inequality between the arithmetic, $\mathcal{A}_2 = (Z_1^c + Z_2^c)/2$, and geometric, $\mathcal{G}_2 = (Z_1\, Z_2)^{c/2}$, mean [Gradshteyn and Ryzhik, 2000, Section 11.116], $\mathcal{A}_2 \geq \mathcal{G}_2$, S can be lower bounded as $S \geq 2\sqrt{\mathcal{P}}$. Using (6), it can be easily seen that the CDF of S can be upper bounded as $F_S(y) \leq F_{\mathcal{P}}[(y/2)^2]$, resulting in (12).

Letting $c = 1$ and for $\rho = 0$, (12) reduces to $F_S(y) \leq 1 - 2\,\lambda_1^{-1/2}\,(y/2)^{\beta} \times K_1\left[2\,\lambda_1^{-1/2}\,(y/2)^{\beta}\right]$, while for $\beta = 2$ simplifies to

$$F_S(y) \leq 1 - \frac{y^2/2}{\sqrt{\lambda_1}\,(1-\rho)} \left\{ I_0\left[\frac{\sqrt{\rho}\,y^2/2}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_1\left[\frac{y^2/2}{\sqrt{\lambda_1}\,(1-\rho)}\right] + \sqrt{\rho}\, I_1\left[\frac{\sqrt{\rho}\,y^2/2}{\sqrt{\lambda_1}\,(1-\rho)}\right] K_0\left[\frac{y^2/2}{\sqrt{\lambda_1}\,(1-\rho)}\right]\right\}. \tag{13}$$

Note that when $\rho = 0$ and $\beta = 2$ ($c = 1$), (12) agrees with a result obtained in [Karagiannidis *et al.*, 2005, eq. (20)]. Also it is worth mentioning that the problem of obtaining an upper bound for the CDF of S with nonidentically distributed RVs, e.g., $\mathcal{W}_{Z_1}(\beta, \Omega_1)$ and $\mathcal{W}_{Z_2}(\beta, \Omega_1)$, may be equivalently stated as finding an upper bound for the CDF of a weighted sum of two identical RVs, e.g. both $\mathcal{W}_{Z_\ell}(\beta, \Omega)$, with weights $w_\ell = \sqrt{\Omega_\ell/\Omega}$.

Having numerically evaluated (12) for $c = 1$, $F_{\mathcal{P}}[(y/2)^2]$ is plotted in Figs. 5 and 6 as a function of y for $\Omega_1 = 1.2$ and $\Omega_2 = 0.7$ ($\lambda_1 = 0.84$). In Fig. 5, $F_{\mathcal{P}}[(y/2)^2]$ is plotted for $\rho = 0.25$ and several values of β, while in Fig. 6, $F_{\mathcal{P}}[(y/2)^2]$ is plotted for $\beta = 2.7$ and two values of ρ. In both figures simulation results for the CDF of $\mathcal{S}$ are also included for comparison purposes. By comparing the numerically evaluated results with the computer

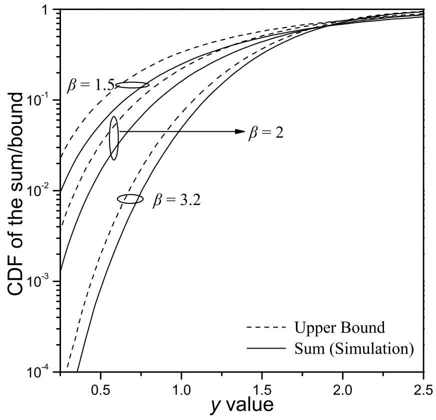

Fig. 5. CDF of an upper bound of the sum of two correlated RVs compared to exact results by simulation ($\rho = 0.25$).

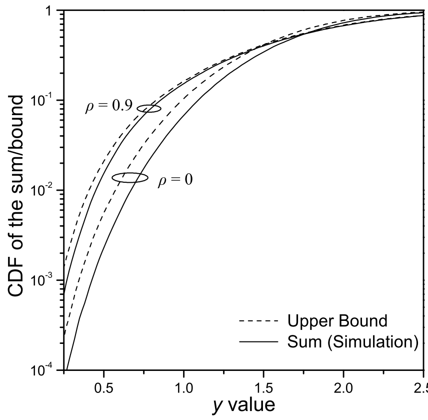

Fig. 6. CDF of an upper bound of the sum of two correlated RVs compared to exact results by simulation ($\beta = 2.7$).

simulated ones, we deduce a close match between them. Specifically, the results clearly show that the higher ρ the tighter the bounds are, while the difference between simulations and bounds slightly decrease as β increases. The trend of the results can be explained as follows. As it is clear, the lower the difference between the left and right hand sides of $\mathcal{A}_2 \geq \mathcal{G}_2$, the tighter the bounds are. In fact, equality holds if and only if $Z_1 = Z_2$. This is the case as β and/or ρ increase (for $\rho \to 1$ and/or $\beta \to +\infty$, $Z_1 \to Z_2$).

5 Conclusions

We derived the distributions of the product and the ratio of two correlated Weibull RVs belonging to a Gaussian class of distributions. A tight upper bound for the distribution of the sum of two correlated Weibull RVs was also presented and it was shown that the higher correlation and/or shaping parameters, the tighter the bounds are.

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