Error Rate Performance of Multilevel Signals with Coherent Detection

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***Abstract*— In this paper, coherent detection for multilevel correlated signaling sets in additive white Gaussian noise is addressed. From the decision rule, a general analytical expression for the symbol error probability (SEP) is derived, which is in the form of a single integral. Our new result is not only in agreement with a known equivalent expression, but it also has a much simpler analytical form. The structure of the correlation matrix under consideration of the signaling set is quite generic, including various known signaling sets as special cases. Hence the derived SEP expression is useful for analyzing the performance of various coherent modulation schemes such as multilevel phase-shift and frequency-shift keying. Specific correlation structures which minimize the SEP are also studied. Based on eigendecomposition or LU decomposition, generic methods for constructing a correlated signaling set for any correlation matrix under consideration are also provided.**

I. Introduction

The rapidly increasing need for high data rates and high quality of services pushes engineers to design systems that will provide optimum performance under a number of parameters and constraints existing in all layers. In the physical layer, there are many options that are available and must be carefully addressed. Some of them with major impact to system performance are the detection scheme, the modulation scheme and order, and the signaling set [1]–[5]. Among them, the choice of the detection scheme is the most significant. As well known, depending on whether or not the receiver is equipped with a phase recovery circuit, the detection scheme can be classified as coherent and noncoherent, with the former providing the best performance among them at the expense of implementation complexity. Although many combinations of the various options are possible, some are more popular than others. A well known combination met in practice is for multilevel frequency-shift keying (FSK) [6], where noncoherent detection with orthogonal signaling is commonly used. However, neither noncoherent detection nor use of an orthogonal signaling set leads to optimum performance. For example, it is well known that in the case of binary FSK (BFSK) modulation, the best performance occurs for coherent reception with the correlation between the two available signals being equal to -0.2172 [1, Section 8.1.1.6], i.e., when the two signals are not orthogonal. This means that orthogonality does not necessarily lead to optimum error performance, while it is obvious that coherent detection always provides better performance than noncoherent. It is therefore reasonable to seek for and study correlated signaling sets in conjunction with coherent receivers in order to minimize the probability of error.

This paper deals with coherent detection in additive white Gaussian noise (AWGN) for signaling sets having a special correlation structure. From the decision rule, a general and simple analytical expression for the symbol error probability (SEP) is derived. The most important advantage of the presented expression is its simplicity compared to a known equivalent expression in the form of multiple integrals. The derived expression can be efficiently used for numerically evaluating the SEP performance for a variety of modulation schemes, such as phase-shift keying (PSK) and FSK. Moreover, the structure of the correlation matrix is quite generic, including various known signaling sets as special cases. In this respect, we study specific correlation structures for signaling sets that minimize the SEP. Based on eigendecomposition or LU decomposition, we also provide generic methods for constructing a specific signaling set for any correlation matrix under consideration.

II. Coherent Detection of Multilevel Signals in AWGN

A. Preliminaries

Let $r(t) \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), denote the received signal of modulation order M at the time instant t, where $0 \leq t \leq T_s$ and T_s is the symbol duration. We can express $r(t)$ as

$$r(t) = s(t) + n(t)\,, \tag{1}$$

where $n(t) \in \mathbb{R}$ is the AWGN with two-sided power spectral density of $N_0/2$, i.e., $\mathbb{E}\langle n(t)\rangle = 0$ and $\mathbb{E}\langle n(t_1)\, n(t_2)\rangle = (N_0/2)\delta(t_1 - t_2)\ \forall t_1, t_2$, with $\mathbb{E}\langle\cdot\rangle$ denoting the expectation operator, $\delta(\cdot)$ denoting the Dirac delta function, and $s(t) = \{s_1(t), s_2(t), \ldots, s_M(t)\} \in \mathbb{R}$ is one of the M signals each having energy $E_i = \int_0^{T_s} s_i^2(t)\,\mathrm{d}t$ ($i = 1, 2, \ldots, M$). All M signals are assumed to have equal a priori probability, but are not necessarily orthogonal to each other, with the correlation coefficient between $s_i(t)$ and $s_j(t)$ ($i, j = 1, 2, \ldots, M$) being

$$\rho_{i,j} = \frac{1}{\sqrt{E_i\, E_j}} \int_0^{T_s} s_i(t)\, s_j(t)\,\mathrm{d}t\,, \tag{2}$$

with $-1 \leq \rho_{i,j} \leq 1$. Moreover, $s_i(t)$ can be expanded as a weighted sum of a set of orthonormal basis functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)$ of dimension $N \leq M$ as

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$s_i(t) = \sum_{j=1}^{N} s_{i,j}\,\varphi_j(t)$, with $\int_0^{T_s} \varphi_j(t)\,\varphi_k(t)\,\mathrm{d}t = \delta[j-k]$, where $\delta[\cdot]$ denotes the Kronecker delta function, and $s_{i,j} = \int_0^{T_s} s_i(t)\,\varphi_j(t)\,\mathrm{d}t$. When the ith signal $s_i(t)$ is transmitted, we can rewrite (1) in vector form as $\underline{r} = \underline{s}_i + \underline{n}$, where $\underline{s}_i = [s_{i,1}, s_{i,2}, \cdots, s_{i,N}]^T$ is the ith signal vector, $(\cdot)^T$ denoting transpose, and $\underline{n}$ is an N-dimensional zero-mean real Gaussian vector with covariance matrix $(N_0/2)\underline{I}_N$, $\underline{I}_N$ denoting the $N \times N$ identity matrix. Thus $\underline{n} \sim \mathcal{N}(\underline{0}_N, (N_0/2)\underline{I}_N)$, with $\underline{0}_N$ denoting the $N \times 1$ vector of zeros.

B. Coherent Detection

From the point of view of detection theory, the detected signal vector $\underline{\hat{s}}_i$ is given by the decision rule

$$\underline{\hat{s}}_i = \underset{\underline{s}_j \in \{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_M\}}{\arg\min} \left|\left|\underline{r} - \underline{s}_j\right|\right|^2 , \quad \text{or}$$

$$\hat{s}_i(t) = \underset{s_j(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\}}{\arg\min} \left[\int_0^{T_s} r(t)\, s_j(t)\mathrm{d}t - \frac{1}{2} E_j \right] . \tag{3}$$

When the ith signal is transmitted, a correct decision occurs under the condition $\underline{\hat{s}}_i = \underline{s}_i$.

We first define the random variable (RV) L_j as $L_j \triangleq \int_0^{T_s} r(t)\, s_j(t)\,\mathrm{d}t - E_j/2$ and then study its statistics. Under the hypothesis that the ith signal is transmitted among the M available, L_j can be expressed as

$$L_j(i) = \begin{cases} \rho_{i,j}\sqrt{E_i\,E_j} - \frac{1}{2}E_j + \int_0^{T_s} n(t)\,s_j(t)\,\mathrm{d}t\,, & i \neq j, \\ \frac{1}{2}E_i + \int_0^{T_s} n(t)\,s_i(t)\,\mathrm{d}t\,, & i = j. \end{cases} \tag{4}$$

It can be easily shown that $L_j(i)$ is a Gaussian RV with $L_j(i) \sim \mathcal{N}(\rho_{i,j}\sqrt{E_i\,E_j} - E_j/2,\;\; E_j\,N_0/2)$. Let us define another vector $\underline{L}(i)$ of dimension M as $\underline{L}(i) \triangleq [L_1(i), L_2(i), \cdots, L_M(i)]^T$, with mean of the jth element as

$$\mathbb{E}\left\langle L_j(i)\right\rangle = \begin{cases} \rho_{i,j}\sqrt{E_i\,E_j} - \frac{1}{2}E_j\,, & i \neq j, \\ \frac{1}{2}E_i\,, & i = j, \end{cases} \tag{5}$$

and covariance between the jth and the kth elements as

$$\mathrm{cov}\left[L_j(i), L_k(i)\right] = \begin{cases} \frac{N_0}{2}\rho_{j,k}\sqrt{E_j\,E_k}\,, & j \neq k, \\ \frac{N_0}{2}E_j\,, & j = k. \end{cases} \tag{6}$$

We can now express the covariance matrix of $\underline{L}(i)$ as

$$\mathrm{cov}\left[\underline{L}(i)\right] = \frac{N_0}{2}\sqrt{\underline{E}}\,\underline{R}_s\sqrt{\underline{E}}\,, \tag{7}$$

where $\sqrt{\underline{E}}$ is a diagonal matrix, the elements of which are the square roots of the symbol energies $E_1, \ldots, E_M$, and $\underline{R}_s$ is the symmetric *correlation matrix* of the signaling set, given by

$$\underline{R}_s = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,M} \\ \rho_{1,2} & 1 & \cdots & \rho_{2,M} \\ \vdots & \vdots & \ddots & \\ \rho_{1,M} & \rho_{2,M} & \cdots & 1 \end{bmatrix} . \tag{8}$$

III. Symbol Error Probability Analysis

Under the hypothesis that the ith signal is transmitted, we find from (3) that the correct decision occurs when $L_i(i)$ is the largest among $L_1(i), \ldots, L_M(i)$. The probability of correct decision is therefore [1, eq. (5-2-17)], [3, eq. (60)]

$$P_{c_i} = \Pr\left[L_j(i) - L_i(i) < 0,\, j \neq i,\, j = 1, 2, \ldots, M\right] . \tag{9}$$

Let $x_j(i) \triangleq L_j(i) - L_i(i)$. Since we need to study the statistics of $x_1(i), \ldots, x_{i-1}(i), x_{i+1}(i), \ldots, x_M(i)$, we define the vector $\underline{x}(i)$ as $\underline{x}(i) \triangleq [x_1(i), \cdots, x_{i-1}(i), x_{i+1}(i), \cdots, x_M(i)]^T$. Using (4), the mean of $x_j(i)$ is given by $\mathbb{E}\langle x_j(i)\rangle = \rho_{i,j}\sqrt{E_i\,E_j} - (E_i + E_j)/2$, and denoting n_j as $n_j = \int_0^{T_s} n(t)\,s_j(t)\,\mathrm{d}t$, the RV $L_j(i)$ can be rewritten as

$$L_j(i) = \begin{cases} \rho_{i,j}\sqrt{E_i\,E_j} - \frac{1}{2}E_j + n_j\,, & i \neq j, \\ \frac{1}{2}E_i + n_i\,, & i = j. \end{cases} \tag{10}$$

It can be easily shown that $\mathbb{E}\langle n_j\rangle = 0$ and

$$\mathbb{E}\left\langle n_j\,n_k\right\rangle = \begin{cases} \frac{N_0}{2}\rho_{j,k}\sqrt{E_j\,E_k}\,, & j \neq k, \\ \frac{N_0}{2}E_j\,, & j = k. \end{cases} \tag{11}$$

For $j, k \neq i$, the covariance between $x_j(i)$ and $x_k(i)$ is therefore given by

$$\begin{aligned} &\mathrm{cov}\left[x_j(i), x_k(i)\right] \\ &= \begin{cases} \frac{N_0}{2}\left(\rho_{j,k}\sqrt{E_j\,E_k} - \rho_{k,i}\sqrt{E_k\,E_i} - \rho_{j,i}\sqrt{E_j\,E_i} + E_i\right), & j \neq k, \\ \frac{N_0}{2}\left(E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i}\right), & j = k. \end{cases} \end{aligned} \tag{12}$$

From (12), the correlation coefficient $\epsilon_{j,k}(i)$ between $x_j(i)$ and $x_k(i)$ is

$$\begin{aligned} &\epsilon_{j,k}(i) = \\ &\frac{\rho_{j,k}\sqrt{E_j\,E_k} - \rho_{k,i}\sqrt{E_k\,E_i} - \rho_{j,i}\sqrt{E_j\,E_i} + E_i}{\sqrt{E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i}}\sqrt{E_k + E_i - 2\,\rho_{k,i}\sqrt{E_k\,E_i}}} . \end{aligned} \tag{13}$$

We now consider the case where, $\forall\, j \neq k$, $\epsilon_{j,k}$ can be written as a product of two separate terms with respect to the indices j and k as

$$\epsilon_{j,k}(i) = \nu_{j,i}\,\nu_{k,i}\,. \tag{14}$$

Let us define the quantities $\Omega_j(i)$ and $\mu_j(i)$ as

$$\Omega_j(i) \triangleq \frac{N_0}{2}\left(E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i}\right) \tag{15a}$$

and

$$\mu_j(i) \triangleq -\rho_{j,i}\sqrt{E_j\,E_i} + \frac{1}{2}\left(E_j + E_i\right) , \tag{15b}$$

respectively. Since $x_j(i)$ is a Gaussian RV, following a standard way of decomposing jointly distributed Gaussian RVs with certain correlation properties into independent and identically distributed (i.i.d.) Gaussian RVs [7], $x_j(i)$ can be expressed as

$$\begin{aligned} x_j(i) &= -\mu_j(i) + n_j - n_i \\ &= -\mu_j(i) + \sqrt{\Omega_j(i)}\left(\sqrt{1-\nu_{j,i}^2}\,\vartheta_j + \nu_{j,i}\,\vartheta_i\right) , \end{aligned} \tag{16}$$

with $\vartheta_1, \vartheta_2, \ldots, \vartheta_M$ being i.i.d. Gaussian RVs, each having a $\mathcal{N}(0,1)$ distribution. By substituting (16) in (9), the probability of correct decision is given by

$$P_{c_i} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right) \times \prod_{\substack{j=1 \\ j\neq i}}^{M} \Pr\left[x_j(i) < 0 | \, \vartheta_i\right] \mathrm{d}\vartheta_i$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right) \times \prod_{\substack{j=1 \\ j\neq i}}^{M} Q\left(\frac{-\mu_j(i) + \nu_{j,i}\,\vartheta_i \sqrt{\Omega_j(i)}}{\sqrt{\Omega_j(i)}\sqrt{1-\nu_{j,i}^2}}\right) \mathrm{d}\vartheta_i\,, \quad (17)$$

with $Q(x) = \mathrm{erfc}(x/\sqrt{2})/2$, erfc $(\cdot)$ being the complementary error function. Based on (17), the SEP is given by $P_{es} = 1 - \sum_{i=1}^{M} P_{c_i}/M$. For $M > 2$, the integral in (17) cannot be expressed in closed form, but it can be easily evaluated via numerical integration using any well known mathematical software package.

The formula given in [2, eq. (4.80)] as for P_{c_i} is

$$P_{c_i} = \int_{-\infty}^{\infty} \int_{-\infty}^{z_i+\sqrt{2\gamma_s}(1-\rho_{i,1})} \cdots \int_{-\infty}^{z_i+\sqrt{2\gamma_s}(1-\rho_{i,j\neq i})} \cdots \times \int_{-\infty}^{z_i+\sqrt{2\gamma_s}(1-\rho_{i,M})} f_{\underline{Z}}\left(\underline{z}\right) \mathrm{d}\underline{z}\,, \quad (18)$$

where $\underline{Z} = [Z_1, Z_2, \cdots, Z_M]^T$ is a vector of M zero-mean jointly Gaussian RVs, $\underline{z} = [z_1, z_2, \cdots, z_M]^T$, $\gamma_s = E_s/N_0$ is the signal-to-noise ratio (SNR) per symbol, and

$$f_{\underline{Z}}\left(\underline{z}\right) = \frac{1}{(2\pi)^{M/2} \det\left|\underline{R}_s\right|^{1/2}} \exp\left(-\frac{1}{2}\,\underline{z}^T\,\underline{R}_s^{-1}\,\underline{z}\right) \quad (19)$$

is the joint probability density function of the elements of $\underline{Z}$. By comparing (17) with (18), it is clear that our new result (17) is much simpler, since it is in the form of a single integral. More importantly, it should be mentioned that the numerical evaluation of (17) is much less time consuming and complicated. Also, (18) is limited only to equienergy signals, i.e., $E_i = E_s \;\forall i$.

Next, some special cases of (17) with practical interest are provided.

A. Binary Signaling

For binary signaling ($M = 2$), $\nu_{j,i}(i) = \nu_{k,i}(i) = 1$ ($i, j, k = 1$ and 2), and by substituting (16) in (17), P_{c_i} reduces to

$$P_{c_i} = Q\left(-\mu_j(i)/\sqrt{\Omega_j(i)}\right)\,. \quad (20)$$

The probability of an erroneous decision is $P_{e_i} = 1 - P_{c_i}$. Using (15), the bit error probability $P_{eb} = (P_{e_1} + P_{e_2})/2$ can be expressed as

$$P_{eb} = Q\left(\sqrt{\frac{E_1 + E_2 - 2\,\rho_{1,2}\sqrt{E_1 E_2}}{2\,N_0}}\right)\,, \quad (21)$$

which is in agreement with [4, eq. (B.33)]. Since $Q(\cdot)$ is a decreasing function of its argument, the minimum value of P_{eb} is obtained when the argument of $Q(\cdot)$ becomes maximum, which occurs for $\rho_{1,2} = \rho_{\min} = -1$.

B. Multilevel Signaling

In order for $\epsilon_{j,k}(i)$ to have the form presented in (14), a general solution exists when

$$\rho_{i,j} = \frac{\rho_i\,E_i + \rho_j\,E_j}{2\sqrt{E_i\,E_j}}\,, \quad (22)$$

where $\rho_i \leq 1$ is a function of the index i, e.g., $\rho_i = \rho^i \;\forall i$. For such a correlation coefficient,

$$\nu_{j,i} = 1/\sqrt{1+\zeta_{j,i}}\,, \quad (23)$$

where $\zeta_{j,i} = (E_j/E_i)\,(1-\rho_j)/(1-\rho_i)$. However, since $-1 \leq \rho_{i,j} \leq 1$, it can be easily shown that the following constraints on ρ_i and ρ_j exist:

$$\rho_i\,E_i + \rho_j\,E_j \leq 2\sqrt{E_i\,E_j}\,,\; \rho_i\,E_i + \rho_j\,E_j \geq -2\sqrt{E_i\,E_j}\,.$$

In addition to these constraints, ρ_i must also satisfy the condition that for $\rho_{i,j} = 0$, $\rho_i = 0 \;\forall i$. An attractive as well as simple solution is $\rho_i = i\,\rho/M$ with $-1 \leq \rho \leq 1$. In this case, the correlation coefficient $\rho_{i,j}$ takes the form

$$\rho_{i,j} = \frac{\rho\,(i\,E_i + j\,E_j)}{2\,M\sqrt{E_i\,E_j}}\,. \quad (24)$$

1) Orthogonal Signaling: When all the M signals are orthogonal to each other, $\rho_{i,j} = 0 \;\forall i \neq j$. This implies $\rho_i = \rho_j = 0$ and $\nu_{j,i} = 1/\sqrt{1+(E_j/E_i)}$. Therefore (17) reduces to

$$P_{c_i} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right) \times \prod_{\substack{j=1 \\ j\neq i}}^{M} Q\left(-\sqrt{\frac{E_i}{2\,N_0}}\sqrt{\frac{E_i}{E_j}} - \sqrt{\frac{E_j}{2\,N_0}} + \sqrt{\frac{E_i}{E_j}}\,\vartheta_i\right) \mathrm{d}\vartheta_i. \quad (25)$$

For $E_i = E_j = E_s \;\forall i,j$ ($P_{c_i} = P_c$), the SEP $P_{es} = 1 - P_c$ agrees with [5, eq. (8.41)].

2) Multilevel Equienergy Signaling: Consider the case of equienergy signaling with $E_i = E_s \;\forall i = 1, 2, \ldots, M$. Now (15) and (22) reduce to

$$\Omega_j(i) = E_s\,N_0\,(1-\rho_{j,i})\,, \quad (26a)$$

$$\mu_j(i) = E_s\,(1-\rho_{j,i})\,, \quad (26b)$$

and

$$\rho_{i,j} = \frac{\rho_i + \rho_j}{2}\,, \tag{27}$$

respectively. Such a correlation structure satisfies (14) with

$$\nu_{j,i} = \sqrt{\frac{1-\rho_i}{2-\rho_i-\rho_j}}\,, \tag{28}$$

while (17) has the form

$$P_{c_i} = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right) \times \prod_{\substack{j=1\\ j\neq i}}^{M} Q\left(\frac{-\sqrt{\frac{E_s}{2\,N_0}}\,(2-\rho_i-\rho_j)+\sqrt{1-\rho_i}\,\vartheta_i}{\sqrt{1-\rho_j}}\right) \mathrm{d}\vartheta_i\,. \tag{29}$$

Note that for orthogonal signaling, i.e., $\rho_i = \rho_j = 0\ \forall i, j$ ($P_{c_i} = P_c$), the SEP $P_{es} = 1 - P_c$ agrees with [5, eq. (8.41)]. For equienergy signals, (24) becomes

$$\rho_{i,j} = \frac{\rho}{2\,M}\,(i+j)\,, \tag{30}$$

showing that $\underline{R}_s$ is a Hankel matrix since each entry of $\underline{R}_s$ is a function of $(i+j)$ [8, Section 4.8.1].

3) Multilevel Equienergy Signaling with Constant Correlation: For the special case of the constant correlation model, we have $\rho_{j,i} = \rho\ \forall j \neq i$. Setting $\rho_i = \rho_j = \rho$ in (28), we get $\epsilon_{j,k}(i) = 1/2$, $\nu_{j,i} = \nu_{k,i} = 1/\sqrt{2}$, and (29) reduces to

$$P_{c_i} = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right) \times Q^{M-1}\left(-\sqrt{\frac{2\,E_s}{N_0}\,(1-\rho)}+\vartheta_i\right) \mathrm{d}\vartheta_i. \tag{31}$$

Since $P_{c_i} = P_c\ \forall i$, the SEP is

$$P_{es} = (M-1)\,Q\left(\sqrt{\frac{E_s}{N_0}\,(1-\rho)}\right) + \frac{1}{\sqrt{2\,\pi}} \sum_{k=2}^{M-1} (-1)^{k+1} \binom{M-1}{k} \times \int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta^2}{2}\right) Q^k\left(\sqrt{\frac{2\,E_s}{N_0}\,(1-\rho)}-\vartheta\right) \mathrm{d}\vartheta. \tag{32}$$

Note that for $M = 2$ and $\rho = 0$, (32) agrees with (21) with $E_1 = E_2 = E_s$ and [5, eq. (8.41)], respectively. It can be shown that P_{es} is an increasing function of ρ, and therefore the minimum value of P_{es} is obtained for $\rho = -1$. However, it is explained in Subsection IV-A that the minimum value for ρ that can be used to form an appropriate signaling set depends on M and is given by

$$\rho_{\min} = -\frac{1}{M-1}\,. \tag{33}$$

For $M = 2$, this result is in agrement with what is presented in Subection III-A.

IV. Generation of Multilevel Signaling Sets

Given a set of M orthonormal signals in complex representation form, we can easily generate a signaling set with any correlation structure. The approach is as follows:

- Let $\varphi_1(t), \varphi_2(t), \ldots, \varphi_M(t)$ denote a set of M orthonormal signals over $[0, T_s)$. Suppose that we want a signaling set $s_1(t), s_2(t), \ldots, s_M(t)$ such that $\int_0^{T_s} s_i(t)\, s_j^*(t)\, \mathrm{d}t = 2\sqrt{E_i\,E_j}\,\rho_{i,j}$, where $(\cdot)^*$ denotes the complex conjugate.
- Let $\underline{\varphi}(t) = [\varphi_1(t), \varphi_2(t), \cdots, \varphi_M(t)]^T$. Note that $\int_0^{T_s} \underline{\varphi}(t)\ \underline{\varphi}^H(t)\, \mathrm{d}t = \underline{I}_M$, where $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operator, owing to orthonormality.
- Let $\underline{s}(t) = [s_1(t), s_2(t), \cdots, s_M(t)]^T$ and let $\underline{R}_s = \int_0^{T_s} \underline{s}(t)\ \underline{s}^H(t)\, \mathrm{d}t$. The element in the ith row and the jth column of $\underline{R}_s$ is $2\sqrt{E_i\,E_j}\,\rho_{i,j}$. Note that $\underline{R}_s$ is a Hermitian matrix.

If we express

$$\underline{s}(t) = \underline{A}\,\underline{\phi}(t)\,, \tag{34}$$

where $\underline{A}\underline{A}^H = \underline{R}_s$, then $\underline{s}(t)$ will have the desired correlation matrix $\underline{R}_s$. Hence, we can have a new signaling set which has the desired correlation structure.

A. Equienergy Signals with Constant Correlation

In the case of equienergy signals with constant correlation coefficient ρ, $\underline{R}_s$ is called the Mth-order intraclass correlation matrix [5, Section 9.7.4.2]. The eigenvalues of $\underline{R}_s$ are $1-\rho$ and $1+(M-1)\rho$, with multiplicities $M-1$ and 1, respectively. Using the well known eigendecomposition method, $\underline{R}_s$ can be decomposed as

$$\underline{R}_s = \underline{U}\,\underline{D}\,\underline{U}^H\,, \tag{35}$$

where $\underline{U}$ is an $M \times M$ unitary matrix containing the orthonormal eigenvectors of $\underline{R}_s$ in its columns and $\underline{D}$ the corresponding diagonal matrix of eigenvalues. The matrix $\underline{A}$ can be obtained as

$$\underline{A} = \underline{U}\,\sqrt{\underline{D}}\,. \tag{36}$$

Since $\underline{R}_s$ is positive semi-definite, the condition of nonnegative eigenvalues of $\underline{R}_s$ yields the result that the minimum probability of error is obtained for $\rho = \rho_{\min} = -1/(M-1)$. An interesting finding is that as the modulation order M increases, orthogonal signaling tends to be the optimum [3, p. 267]. We must also mention that instead of using the eigendecomposition method, a known class of signals, namely simplex signals, can be constructed. These signals are equienergy, equicorrelated, and satisfy (33) [1, eq. (4-3-35)].

Next, for the specific cases of $M = 2$ and 4, we present the matrix $\underline{A}$ for which the SEP is minimized.

1) Binary Signaling: In the case of binary signaling ($M = 2$), the minimum correlation coefficient is given by $\rho = -1$ and

$$\underline{U} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (37)$$

From (36), the matrix $\underline{A}$ is given by

$$\underline{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}. \quad (38)$$

This implies, from (34), that the new signaling set consists of two antipodal signals, i.e., binary PSK.

2) Quadrature Signaling: In the case of quadrature signaling ($M = 4$), the minimum correlation coefficient is given by $\rho = -1/3$ and

$$\underline{U} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}. \quad (39)$$

From (36), the matrix $\underline{A}$ is given by

$$\underline{A} = \begin{bmatrix} -\frac{2}{\sqrt{6}} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ \frac{2}{\sqrt{6}} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \end{bmatrix}. \quad (40)$$

B. Signals with Arbitrary Correlation

When $\underline{R}_s$ does not have a constant correlation structure, it is very difficult to derive the eigenvalues and the eigenvectors of $\underline{A}$ in closed form using the eigendecomposition method. Thus, as a general case, the LU decomposition method [9] can be followed. In this method, we can split $\underline{R}_s$ as

$$\underline{R}_s = \underline{L}\,\underline{D}'\,\underline{L}^H, \quad (41)$$

where all the main diagonal entries of the lower triangular matrix $\underline{L}$ are equal to one ($\underline{L}^H$ is upper triangular) and each of the main diagonal entries of the diagonal matrix $\underline{D}'$ is equal to the corresponding leading principal minor of $\underline{R}_s$. The matrix $\underline{A}$ can be obtained as

$$\underline{A} = \underline{L}\sqrt{\underline{D}'}. \quad (42)$$

V. Numerical Results

As an indicative example of the applicability of (17), Fig. 1 shows the SEP of equienergy signals as a function of the SNR per symbol. A constant correlation model is considered with the correlation coefficient given by (33) for $M = 2$, 4, and 8. Three corresponding curves for the SEP of orthogonal signaling are also included for comparison. Interestingly, the three curves corresponding to equicorrelated signals indicate a better error performance than those corresponding to orthogonal signaling. More specifically, for about $P_{es} \leq 10^{-3}$, a constant difference in the SNR between equicorrelated and orthogonal signaling, which is $10\,\log_{10}[M/(M-1)]$, is observed [1, eq. (5-2-35)]. However, when M increases, this difference tends to vanish.

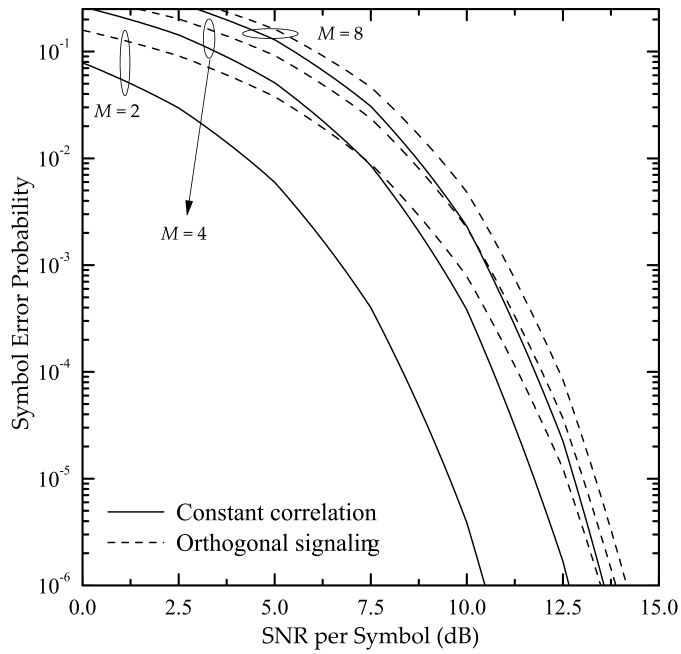

Fig. 1. Comparison between the SEP of multilevel correlated (constant correlation with $\rho = -1/(M-1)$) and orthogonal signals with coherent detection as a function of the received SNR per symbol.

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