The $N*$Nakagami Fading Channel Model

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***Abstract*— A generic distribution, referred to as $N*$Nakagami, constructed as the product of N statistically independent, but not necessarily identically distributed, Nakagami-m random variables (RV)s, is introduced and analyzed. The proposed distribution turns out to be an extremely convenient tool for analyzing the performance of digital communication systems over generalized fading channels. The main result contributions of the paper are two-fold. Firstly, the moments-generating function (MGF), probability density function, cumulative distribution function (CDF), and moments of the $N*$Nakagami distribution are derived in closed-form. Using these formulae, generic closed-form expressions for the outage probability, amount of fading, and average symbol error probability for several binary and multilevel modulation signals of digital communication systems operating over the $N*$Nakagami fading channel model are presented. Various numerical and computer simulation results verify the correctness of the proposed formulation. Secondly, the suitability of the $N*$Nakagami fading distribution to approximate the Lognormal distribution is being investigated. Using Kolmogorov-Smirnov tests, the rate of convergence of the central limit theorem as pertaining to the multiplication of RVs is quantified.**

I. INTRODUCTION

Radio signals generally propagate according to the mechanisms of reflection, diffraction, and scattering, which roughly characterize the radio propagation by three nearly independent phenomena: Path loss variance with distance, shadowing (or long-term fading), and multipath (or short-term) fading. Except path loss, which is only distance dependent, the other two phenomena can be statistically described by fading models where their parameters can be determined by using outputs of experimental radio propagation measurements. These channel models find use in the design and pretest evaluation of wireless communications systems in general and of fading mitigation techniques in particular. As expectations for the performance and reliability of wireless systems become more demanding, the significance of accurate channel modelling in system design, evaluation, and deployment will continue [1].

Due to the existence of a great variety of fading environments, several statistical distributions have been proposed for channel modeling of fading envelopes under short- and long-term fading conditions. Short-term fading models include the well-known Rayleigh, Weibull, Rice, and Nakagami-m [2]–[4] distributions. For long-term fading conditions, it is widely accepted that the probability density function (PDF) of the fading envelopes can be modelled by the well-known Lognormal distribution [5], [6].

Recently, attention has been given to the so-called "multiplicative" fading models [7]. Such models do not separate the fading in several parts but rather study the phenomenon as whole. A physical interpretation for these models is justified by considering a received signal that is generated from the product of a large number of rays reflected via N statistically independent scatterers [7]. For $N = 2$, the so-called double Rayleigh (i.e., Rayleigh×Rayleigh) fading model has been found to be suitable when both the transmitter and receiver are moving [8], while for $N > 1$, Coulson *et al.* have studied the distribution of the product of N correlated Rayleigh distributed RVs via computer simulations [9]. It is interesting to note that the double Rayleigh model has been recently used for keyhole channel modeling of multiple-input multiple-output (MIMO) systems [10], [11]. Extending this model by characterizing the fading between each pair of the transmit and receive antennas in the presence of the keyhole as Nakagami-m, the double Nakagami-m (i.e., Nakagami$-m$ × Nakagami$-m$) fading model has also been considered [12].

In an effort to generalize all previously mentioned research works, in this paper, we introduce and analyze the $N*$Nakagami distribution constructed as the product of N statistically independent, but not necessarily identically distributed, Nakagami-m RVs. First, the statistics of this distribution is studied deriving its moments-generating function (MGF), PDF, CDF, and moments in closed-form. Considering the proposed distribution as the fading channel model of a digital communication system, closed-form expressions are derived for the amount of fading (AoF), outage probability (OP), and average symbol error probability (ASEP) for several families of binary and multilevel modulation formats. Moreover, with the aid of the central limit theorem (CLT), the convergence rate of the proposed model towards the Lognormal distribution with increasing N is investigated.

II. THE $N*$NAKAGAMI DISTRIBUTION: DEFINITION AND STATISTICAL CHARACTERISTICS

Let us consider $N \geq 1$ independent Nakagami[1] distributed RVs $\{R_\ell\}_{\ell=1}^{N}$, each with PDF

$$f_{R_\ell}(r) = \frac{2\, m_\ell^{m_\ell}}{\Omega_\ell^{m_\ell}\, \Gamma(m_\ell)}\, r^{2\,m_\ell-1} \exp\left(-\frac{m_\ell}{\Omega_\ell}\, r^2\right) \qquad (1)$$

[1]From now on, for the conciseness of the presentation the suffix "-m" will be omitted.

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where $\Gamma(\cdot)$ is the Gamma function [13, eq. (8.310/1)], $\Omega_\ell = \mathcal{E}\langle R_\ell^2\rangle$, $m_\ell = \Omega_\ell^2/\mathcal{E}\langle (R_\ell^2 - \Omega_\ell)^2\rangle \geq 1/2$, and $\mathcal{E}\langle\cdot\rangle$ denotes expectation. As well-known, the PDF in (1) includes the cases of Rayleigh ($m_\ell = 1$) and one-sided Gaussian ($m_\ell = 1/2$) distributions as special cases.

Definition 1 (The $N*$Nakagami distribution): The RV Y is defined as the distribution of the product of N independent, but not necessarily identically distributed, RVs R_ℓ, i.e.,

$$Y \triangleq \prod_{i=1}^{N} R_i \tag{2}$$

as the $N*$Nakagami distribution.

Theorem 1 (Moments-generating function): The MGF of Y is given by

$$\mathcal{M}_Y(s) = \frac{1/\sqrt{\pi}}{\prod_{i=1}^{N}\Gamma(m_i)} G_{2,N}^{N,2}\left[\frac{4}{s^2}\prod_{i=1}^{N}\left(\frac{m_i}{\Omega_i}\right)\Big|_{m_1,m_2,\ldots,m_N}^{1/2,1}\right] \tag{3}$$

where $G[\cdot]$ is the Meijer's G-function [13, eq. (9.301)].

Lemma 1 (Probability density function): The PDF of Y is given by

$$f_Y(y) = \frac{2/y}{\prod_{i=1}^{N}\Gamma(m_i)} G_{0,N}^{N,0}\left[y^2\prod_{i=1}^{N}\frac{m_i}{\Omega_i}\Big|_{m_1,\, m_2,\ldots,\, m_N}^{-}\right]. \tag{4}$$

Proof: By applying the inverse Laplace transform $\mathcal{L}^{-1}(\cdot;\cdot)$ to (3), the PDF of Y [13, Section 17.11] $f_Y(y) = \mathcal{L}^{-1}\{\mathcal{M}_Y(s);y\}$ can be expressed in closed-form using [14, eq. (21)] as shown in (4). ■

Lemma 2 (Cumulative distribution function): The CDF of Y is given by

$$F_Y(y) = \frac{1}{\prod_{i=1}^{N}\Gamma(m_i)} G_{1,N+1}^{N,\,1}\left[y^2\prod_{i=1}^{N}\frac{m_i}{\Omega_i}\Big|_{m_1,m_2,\ldots,m_N,0}^{1}\right]. \tag{5}$$

Proof: Since the CDF of Y is given by $F_Y(y) = \int_0^y f_Y(x)\,dx$ by substituting (4) and using [14, eq. (26)], (5) can be obtained. ■

Lemma 3 (Moments): The nth order moment of Y is given by

$$\langle Y^n\rangle = \prod_{i=1}^{N}\frac{\Gamma(m_i + n/2)}{\Gamma(m_i)}\left(\frac{\Omega_i}{m_i}\right)^{n/2}. \tag{6}$$

Proof: Using (2), the nth order moment of Y can be expressed as $\mathcal{E}\langle Y^n\rangle = \mathcal{E}\left\langle\prod_{i=1}^{N} R_i^n\right\rangle$. Since RVs R_ℓ are mutual independent, the above equation can be expressed as the product of the nth order moment of each RV as shown in (6). ■

III. Performance Analysis and Evaluation

Let us consider a digital communication system operating over the $N*$Nakagami fading channel also in the presence of additive white Gaussian noise (AWGN). The instantaneous SNR per symbol at the input of its receiver is given by [2]

$$\gamma = \frac{E_s}{N_0}Y^2 \tag{7}$$

where E_s is the transmitted symbol's average energy and N_0 is the single-sided AWGN power spectral density. The corresponding average SNR is

$$\overline{\gamma} = \mathcal{E}\langle Y^2\rangle\frac{E_s}{N_0} = \frac{E_s}{N_0}\prod_{i=1}^{N}\Omega_i. \tag{8}$$

Dividing (7) and (8) by parts and using (5), the CDF of γ can be derived as

$$F_\gamma(\gamma) = \frac{1}{\prod_{i=1}^{N}\Gamma(m_i)} G_{1,N+1}^{N,\,1}\left[\frac{\gamma}{\overline{\gamma}}\prod_{i=1}^{N} m_i\Big|_{m_1,\, m_2,\ldots,\, m_N,\, 0}^{1}\right]. \tag{9}$$

By taking the first derivative of (9) with respect to γ, the corresponding PDF can be obtained as

$$f_\gamma(\gamma) = \frac{1/\gamma}{\prod_{i=1}^{N}\Gamma(m_i)} G_{0,N}^{N,0}\left[\frac{\gamma}{\overline{\gamma}}\prod_{i=1}^{N} m_i\Big|_{m_1,\, m_2,\ldots,\, m_N}^{-}\right]. \tag{10}$$

Note, that for $N = 1$ and by using [14, eq. (11)], (10) simplifies to [2, eq. (2.7)], while for $N = 2$ and by using [13, eq. (9.34/3)], (10) can be simplified to a previously known result [12, eq. (31)]. With the aid of (7) and (8), the nth moment of γ can be easily derived as

$$\mathcal{E}\langle\gamma^n\rangle = \overline{\gamma}^n\prod_{i=1}^{N}\frac{\Gamma(m_i+n)}{\Gamma(m_i)\,m_i^n}. \tag{11}$$

A. Amount of Fading

The AoF is defined as the ratio of the variance to the square average SNR per symbol, i.e., $A_F \triangleq \mathrm{var}(\gamma)/\overline{\gamma}^2$. Using (11), A_F can be easily expressed in a simple closed-form expression as

$$A_F = \prod_{i=1}^{N}\left(1 + \frac{1}{m_i}\right) - 1. \tag{12}$$

From the above equation, it may be concluded that, since $m_\ell \geq 1/2$, then $0 < A_F \leq 3^N - 1$.

B. Outage Probability

The outage probability, P_{out}, is defined as the probability that the received SNR per symbol falls below a given threshold γ_{th}. Using (9), this probability can be obtained as

$$P_{out}(\gamma_{th}) = F_\gamma(\gamma_{th}). \tag{13}$$

Having numerically evaluated (13), in Fig. 1, P_{out} is plotted as a function of the normalized outage threshold, $\gamma_{th}/\overline{\gamma}$, for the $N*$Nakagami channel with $N = 4$, 8, and 12, and for different values of $m = m_\ell$. These results clearly show that for a fixed value of N, P_{out} improves with an increase of m. This occurs due to the fact that as m increases, the fading severity of the cascaded channels decreases, and hence the deep fading generated by the product of Nakagami fading envelopes also decreases. The obtained results further indicate that for a given value of m there exists a threshold for $\gamma_{th}/\overline{\gamma}$ above (below) which P_{out} improves (degrades) with increasing N. For example, for $m = 2$ this threshold is around 15 dB. Although not shown in Fig. 1, similar thresholds exist for other values of m.

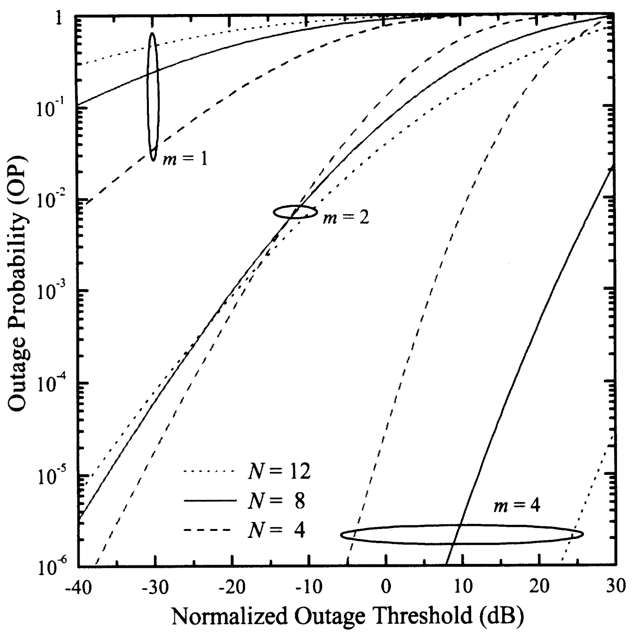

Fig. 1. OP of a digital receiver operating over the $N*$Nakagami fading channel model.

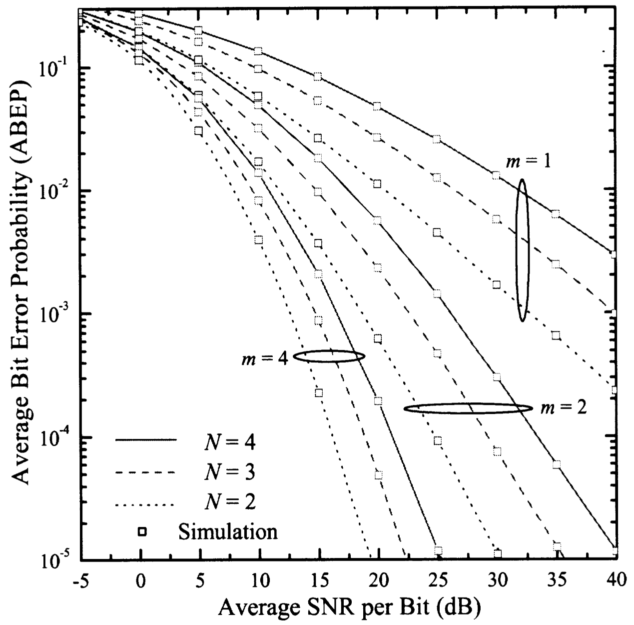

Fig. 2. ABEP performance of Gray encoded QPSK of a digital receiver operating over the $N*$Nakagami fading channel model.

C. Average Symbol Error Probability

The most straightforward approach to obtain the ASEP, $\overline{P}_{se}$, is to average the conditional symbol error probability $P_{se}(\gamma)$ over the PDF of γ, i.e.,

$$\overline{P}_{se} = \int_0^\infty P_{se}(\gamma)\, f_\gamma(\gamma)\, d\gamma. \tag{14}$$

For $P_{se}(\gamma)$ there are well-known generic expressions for different sets of modulation schemes, including:

i) Binary phase/frequency shift keying (BPSK/BFSK) and for higher values of average SNR, differentially encoded BPSK (DEBPSK), quadrable phase shift keying (QPSK), minimum shift keying (MSK), and square M-ary quadrature amplitude modulation (M-QAM), with $M \geq 4$, in the form of $P_{se}(\gamma) = A\,\mathrm{erfc}\left(\sqrt{B\,\gamma}\right)$ where $\mathrm{erfc}(\cdot)$ is the well-known complementary error function [13, eq. (8.250/4)];

ii) Non-coherent BFSK (NBFSK) and binary differential phase shift keying (BDPSK), in the form of $P_{se}(\gamma) = A\,\exp(-B\,\gamma)$;

In the above $P_{se}(\gamma)$ expressions the particular values of A and B depend on the specific modulation scheme. Next, (14) is solved in closed-form for each one of the above sets of signals for the $N*$Nakagami fading channel.

Using (10) and (14), it can be easily recognized that for the first set of modulated schemes (i.e., BPSK, BFSK, DEBPSK, QPSK, MSK, and square M-QAM), the evaluation of definite integrals, which include Meijer's, power, and exponential functions, is required. Since such integrals are not tabulated, the solution can be found with the aid of [14, eq. (21)], and thus the ASEP can be expressed in a closed-form as

$$\overline{P}_{se}(\overline{\gamma}) = \frac{A\,\pi^{-1/2}}{\prod_{i=1}^{N}\Gamma(m_i)}\, G_{2,N+1}^{N,2}\left[\frac{\prod_{i=1}^{N} m_i}{B\,\overline{\gamma}}\,\middle|\,\begin{matrix}1/2\,,\,1\\ m_1, m_2, \ldots, m_N, 0\end{matrix}\right]. \tag{15}$$

Similarly, for the second set (i.e., NBFSK and BDPSK), the ASEP can be derived as

$$\overline{P}_{se}(\overline{\gamma}) = \frac{A}{\prod_{i=1}^{N}\Gamma(m_i)}\, G_{1,N}^{N,1}\left[\frac{\prod_{i=1}^{N} m_i}{B\,\overline{\gamma}}\,\middle|\,\begin{matrix}1\\ m_1, m_2, \ldots, m_N\end{matrix}\right]. \tag{16}$$

Using (15) and (16), the ASEP of various coherent and non-coherent binary and multilevel modulation schemes can be evaluated. As a typical example, in Fig. 2, the average bit error probability (ABEP) of Gray encoded QPSK $\overline{P}_{be} = \overline{P}_{se}/\log_2(M)$ $(M = 4)$ is presented, as a function of the average SNR per bit, $\overline{\gamma}_b = \overline{\gamma}/\log_2(M)$, for several values of $m_\ell = m$ and N. As expected, the performance evaluation results show that $\overline{P}_{be}$ improves with an increase of m and/or a decrease of N. This happens due to the fact that as m increases and/or N decreases, the probability that any of the cascaded fading channels be in deep fade increases significantly. Thus, the higher m and/or lower N are, the lower is the fading severity of the channel. The above analytical performance evaluation results have also been verified by means of computer simulations as illustrated in Fig. 2. It is also noted that similar behavior has been also observed for P_{out} (see Fig. 1).

IV. Lognormal Distribution Approximation

In this section, using statistical tools and arguments, an accurate approximation for the Lognormal distribution with the proposed $N*$Nakagami distribution is presented. In particular, by performing K-S tests, the convergence rate of CLT for the $N*$Nakagami towards the Lognormal distribution is investigated. Furthermore, since to the best of our knowledge, the Lognormal MGF has never been derived in closed-form, a useful and convenient semi-analytical closed-form approximation for this MGF is proposed.

A. Problem Statement and Preliminaries

Let μ_Υ and σ_Υ^2 be the mean and the variance of a Lognormally distributed RV X, respectively, having CDF

$$F_X(x) = 1 - \frac{1}{2}\operatorname{erfc}\left[\frac{\ln(x) - \mu_\Upsilon}{\sqrt{2}\,\sigma_\Upsilon}\right]. \quad (17)$$

The average SNR per symbol, $\overline{\gamma} = \mathcal{E}\left\langle X^2\right\rangle E_s/N_0$, is given by $\overline{\gamma} = \exp\left(\mu_\Upsilon + \sigma_\Upsilon^2/2\right)$.

Our purpose thus is to investigate the necessary and sufficient conditions for the $N*$Nakagami model to become identical to the Lognormal distribution, i.e., $Y \equiv X$. For N large and by applying the CLT, the RV $\Upsilon = \ln(Y) = \sum_{i=1}^{N} \ln(R_i)$ tends towards the Normal (Gaussian) distribution, and consequently Y tends to the Lognormal distribution [15, pp. 220–221]. For $Y \equiv X$, both distributions should have the same mean $\mu_\Upsilon = \mathcal{E}\left\langle \ln(Y)\right\rangle$ and variance $\sigma_\Upsilon^2 = \mathcal{E}\left\langle \ln^2(Y)\right\rangle - \mathcal{E}\left\langle \ln(Y)\right\rangle^2$. Hence, using (2) and [13, eqs. (4.352/1) and (4.358/2)], this mean and variance can be determined as

$$\mu_\Upsilon = \frac{1}{2}\sum_{i=1}^{N}\left[\Psi(m_i) - \ln\left(\frac{m_i}{\Omega_i}\right)\right] \quad (18)$$

and

$$\sigma_\Upsilon^2 = \frac{1}{4}\sum_{i=1}^{N}\Psi^{(1)}(m_i) \quad (19)$$

respectively, where $\Psi^{(1)}(\cdot)$ is the first derivative of the Digamma function $\Psi(\cdot)$ [13, eq. (8.360)].

B. K-S Goodness-of-Fit Tests

In order to measure the difference between two CDFs, a number of statistical tests, such as the absolute value of the area between them, or their integrated mean square difference, may be applied. However, a particularly simple and computational efficient measure, defined as the maximum value of the absolute difference between the two CDFs of X and Y, is the K-S statistical test [15, pp. 272–273]. Thus, for comparing one data set from $F_Y(\cdot)$ to the known $F_X(\cdot)$, the K-S statistical test is defined as

$$\mathcal{T} \triangleq \max\left|F_X(x) - F_Y(x)\right|. \quad (20)$$

Definition 2 (Null hypothesis $\mathbf{H}_0$*):* We define $\mathbf{H}_0$ as the null hypothesis that observed data of Y belong to the CDF of the Lognormal distribution.

To test $\mathbf{H}_0$ that observed data of Y belong to analytical CDF $F_X(\cdot)$, the K-S goodness-of-fit test compares $\mathcal{T}$ to a critical level $\mathcal{T}_{\max}$, as a function of N and significance level α. Any hypothesis for which $\mathcal{T} > \mathcal{T}_{\max}$ is rejected with significance

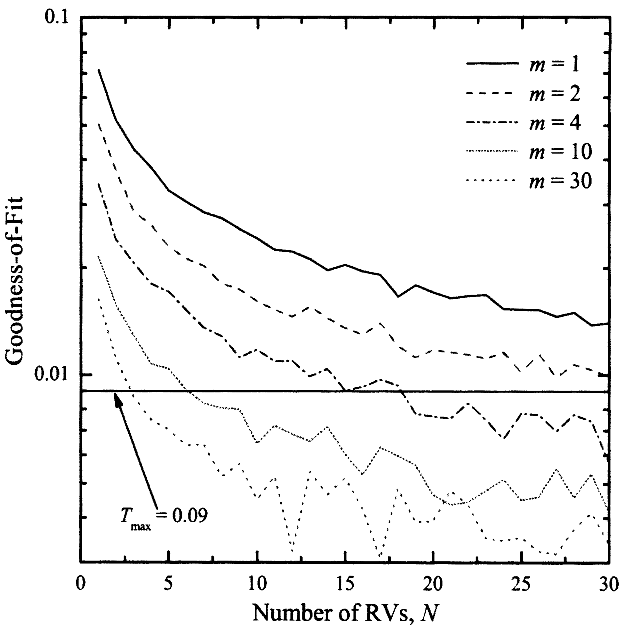

Fig. 3. Hypothesis testing distributions of the product of Nakagami RVs. Comparison of K-S tests (for $\mathbf{H}_0$: Y be Lognormal) referred to 5% significance level.

$1-\alpha$, while any hypothesis for which $\mathcal{T} < \mathcal{T}_{\max}$ is accepted with the same level of significance.

Results following next compare values of $\mathcal{T}$, calculated using (20), against $\mathcal{T}_{\max} = 0.09$ for 10^4 samples and significance level $\alpha = 5\%$ [9, Table I]. Following [9], but without loss of generality, we consider that RVs R_ℓ are i.i.d. Nakagami RVs ($\Omega_\ell = \Omega$ and $m_\ell = m$). Fig. 3 tests $\mathbf{H}_0$ where values of $\mathcal{T}$ are plotted for the data tested against the number of RVs N. Note, that similarly to [9], these results have been obtained by averaging the results of 30 simulation runs, each for 10^4 samples. Extensive simulations have shown that for $m \leq 2$ and $N < 30$, $\mathbf{H}_0$ is rejected with 95% significance although the distribution is clearly converging towards the Lognormal distribution with increasing N, which agrees with the observations made in [7], [9]. However, as m and/or N increase, Y converges for relatively low values of N towards the Lognormal distribution. For example, when $m \geq 10$ and $N \geq 7$, $\mathbf{H}_0$ is accepted with 95% significance.

V. Conclusions

A novel generic distribution, referred to as $N*$Nakagami, constructed as the product of N statistically independent, but not necessarily identically distributed, Nakagami RVs, was introduced and analyzed. Based on this useful distribution, several open research problems was addressed. Firstly, it was used for the performance study of a digital communication systems employing various families of modulation schemes operating over the $N*$Nakagami fading channel model. Secondly, by performing K-S tests, a quantification of the convergence rate of the CLT demonstrated that even for small N, the proposed distribution may accurately and fast approximate the Lognormal distribution and that interestingly, the convergence rate increases with an increase of m. In summary, the $N*$Nakagami distribution is not only useful because it may be used for modeling of fading channels, but also allows the accurate approximation of the mathematically intractable Lognormal distribution. It would be interesting to conduct experimental channel measurements which can verify the suitability of the proposed $N*$Nakagami distribution to indeed model realistic wireless fading channels.

ACKNOWLEDGMENTS

This work has been performed within the framework of the Satellite Network of Excellence (SatNEx) project, a Network of Excellence (NoE) funded by European Committee (EC) under the FP6 program.

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