Distributions Involving Correlated Generalized Gamma Variables

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Abstract. In this paper, some of the most important statistical properties concerning the product and the ratio of two correlated generalized Gamma (GG) random variables (rvs) are studied. The probability density function of both the product and the ratio of two correlated GG rvs are obtained in closed form, while the cumulative distribution function of the product is derived in terms of an infinite series. Capitalizing on the distribution of the product, a union upper bound for the distribution of the sum of two correlated GG rvs is also derived.
Keywords: correlated statistics, distribution of product, distribution of ratio, generalized Gamma, stochastic models.

1 Introduction

The generalized Gamma (GG) distribution, introduced by Stacy in 1962 [Stacy, 1962], has been considered as a generalization of the Gamma distribution. It is a versatile and generic distribution, while it is especially attractive because it incorporates several well-known distributions as special cases, i.e., Rayleigh, Nakagami-m [Nakagami, 1960], and Weibull [Weibull, 1951], as well as the well-known lognormal as a limiting case. The GG distribution has been used in many scientific fields including life data, speech recognition, ultrasonic backscatter signals modeling [Raju and Srinivasan, 2002], [Dat *et al.*, 2005], and [Heard and Pensky, 2006]. More recently, the GG distribution has been considered in wireless communications theory for accurately modeling short term fading in conjunction with long term fading (shadowing) channels [Coulson *et al.*, 1998].

Representative previously published works concerning distribution of ratios, products can be found in [Mathai, 1972], [Nadarajah and Kotz, 2006], [Nadarajah and Gupta, 2005], [Nakagami and Ōta, 1957], [Lee *et al.*, 1979], [Simon, 2002], and [Malik and Trudel, 1986]. Specifically in [Mathai, 1972], [Nadarajah and Kotz, 2006], and [Nadarajah and Gupta, 2005], independent random variables (rv)s have been considered. Moreover, in [Simon, 2002], [Malik and Trudel, 1986], [Lee *et al.*, 1979], and [Nakagami and Ōta, 1957], correlated rvs have been considered. The bivariate Rayleigh, Nakagami, Weibull, and lognormal distributions have been thoroughly studied in the past, e.g., see [Alouini and Simon, 2002] and [Tan and Beaulieu, 1997]. However, the bivariate GG distribution has been introduced only very recently[1] [Piboongungon *et al.*, 2005] and [Yacoub, 2007]. In those papers, the bivariate model has been derived and applied to model fading channels with arbitrary correlation between them. However, to the best of the authors' knowledge, the topic of distributions of products and ratios of correlated GG rvs has not been addressed in the open technical literature, and this is the subject of our paper.

Let $X_\ell \geq 0$ ($\ell = 1$ and 2) represent two correlated GG distributed rvs, which are not necessarily identically distributed, having the joint probability density function (pdf) given by [Piboongungon *et al.*, 2005, eq. (5)]

$$
\begin{aligned}
f_{X_1,X_2}(x_1,x_2) = {} & \frac{4\,\beta^2\, m^{m+1}\,(x_1\,x_2)^{\beta(m+1)-1}\,\rho^{(1-m)/2}}{(\alpha_1\,\alpha_2)^{(m+1)/2}\,(1-\rho)\Gamma(m)} \\
& \times \exp\left[-\frac{m}{1-\rho}\left(\frac{x_1^{2\,\beta}}{\alpha_1}+\frac{x_2^{2\,\beta}}{\alpha_2}\right)\right] I_{m-1}\left[\frac{2m\,(x_1\,x_2)^{\beta}}{1-\rho}\sqrt{\frac{\rho}{\alpha_1\,\alpha_2}}\right].
\end{aligned} \tag{1}
$$

In the above equation $\beta > 0$ and $m \geq 1/2$ are the distribution shaping parameters, $\alpha_\ell = \mathbb{E}\langle X_\ell^2\rangle > 0$ is the scaling parameter (with $\mathbb{E}\langle\cdot\rangle$ denoting expectation), $I_{m-1}(\cdot)$ is the $(m-1)$th order modified Bessel of the first kind [Gradshteyn and Ryzhik, 2000, eq. (8.406)], and $\Gamma(\cdot)$ is the Gamma function [Gradshteyn and Ryzhik, 2000, eq. (8.310/1)]. Moreover, the correlation coefficient between X_1^2 and X_2^2 is related with the correlation coefficient between Nakagami-m rvs as [Piboongungon *et al.*, 2005, eq. (9)]

$$
\varrho = \frac{{}_2F_1(-1/\beta,-1/\beta;m;\rho)-1}{-1+\Gamma(m+2/\beta)\Gamma(m)/\Gamma^2(m+1/\beta)} \tag{2}
$$

with $0 \leq \varrho < 1$ and ${}_pF_q(\cdot)$ representing the generalized hypergeometric function with p, q integers [Gradshteyn and Ryzhik, 2000, eq. (9.14/1)].

In this paper, capitalizing on (1), important statistical metrics, such as the pdf and the cumulative distribution function (cdf) of the product of two correlated GG rvs are derived. Moreover, using an inequality between the

[1] It is clarified that the bivariate GG distribution under consideration is originated by correlated Gaussian rvs.

arithmetic and geometric mean, a useful union upper bound for the distribution of the sum of two correlated GG rvs is also presented. Finally, the pdf of the ratio of two correlated GG rvs is obtained in closed form.

The rest of this paper is organized as follows. After this introduction, in Section 2, the distribution of the product and a bound of the distribution of the sum of two correlated GG rvs are presented. In Section 3, the pdf of the ratio is derived in closed form, while in Section 4, concluding remarks are provided.

2 Product and sum distributions

In this section, based on (1), the pdf and the cdf of $X_1\, X_2$ are obtained, while a union upper bound for the distribution of $X_1 + X_2$ is also extracted.

2.1 Distribution of the product

Let $\mathcal{R}$ be a rv defined as

$$\mathcal{R} \stackrel{\triangle}{=} X_1\, X_2. \tag{3}$$

By applying (1) and (3) in[Papoulis, 2001, eq. (6-74)], changing variables, using [Gradshteyn and Ryzhik, 2000, eq. (3.471/9)], and after some straightforward mathematical manipulations, the pdf of $\mathcal{R}$ is given in closed form by

$$\begin{aligned} f_{\mathcal{R}}(x) = & \frac{4\,\beta\, m^{m+1}\, x^{\beta(m+1)-1}}{(1-\rho)\,\rho^{(m-1)/2}\,\Gamma(m)\,\lambda_1^{(m+1)/2}} \\ & \times I_{m-1}\left(\frac{2m}{1-\rho}\sqrt{\frac{\rho}{\lambda_1}}\,x^{\beta}\right) K_0\left(\frac{2m}{1-\rho}\,\frac{x^{\beta}}{\sqrt{\lambda_1}}\right) \end{aligned} \tag{4}$$

where $\lambda_1 = \alpha_1\,\alpha_2$ and $K_n(\cdot)$ represents the nth-order modified Bessel of the second kind [Gradshteyn and Ryzhik, 2000, eq. (8.407)] ($n \in \mathbb{N}$). By setting $\beta = 1$, (4) reduces to a previously derived result [Nakagami, 1960, eq. (144)].

By integrating (4) with respect to x, and after making a change of variables, an integral of the form $\int_0^A x^m\, I_{m-1}(B\,x)\, K_0(C\,x)\,\mathrm{d}x$ needs to be solved, with $A, B, C > 0$. This type of integral is very difficult, if not impossible, to be solved in closed form. An alternative and mathematically more tractable solution is to employ an infinite series representation for $I_{m-1}(\cdot)$ [Gradshteyn and Ryzhik, 2000, eq. (8.445)]. Following this approach, using [Wolfram, 2006, eq. (03.04.21.0009.01)], and after some mathematical manipulations, the cdf of $\mathcal{R}$ can be obtained as

$$\begin{aligned} F_{\mathcal{R}}(x) = & \sum_{k=0}^{\infty} \frac{2\,\rho^k\, m^{2(k+m)}\, x^{2(k+m)\beta}}{(1-\rho)^{2k+m}\,\Gamma(m)\Gamma(m+k)\,k!\,(k+m)^2\lambda_1^{k+m}} \\ & \times \left[(k+m)\,{}_1F_2\left(1; k+m+1, k+m; \xi^2\, x^{2\beta}\right) K_0\left(2\,\xi\, x^{\beta}\right)\right. \\ & \left. + \xi\, x^{\beta}\, K_1\left(2\,\xi\, x^{\beta}\right) {}_1F_2\left(1; k+m+1, k+m+1; \xi^2\, x^{2\beta}\right)\right] \end{aligned} \tag{5}$$

with $\xi = m\,(1-\rho)^{-1}\lambda_1^{-1/2}$.

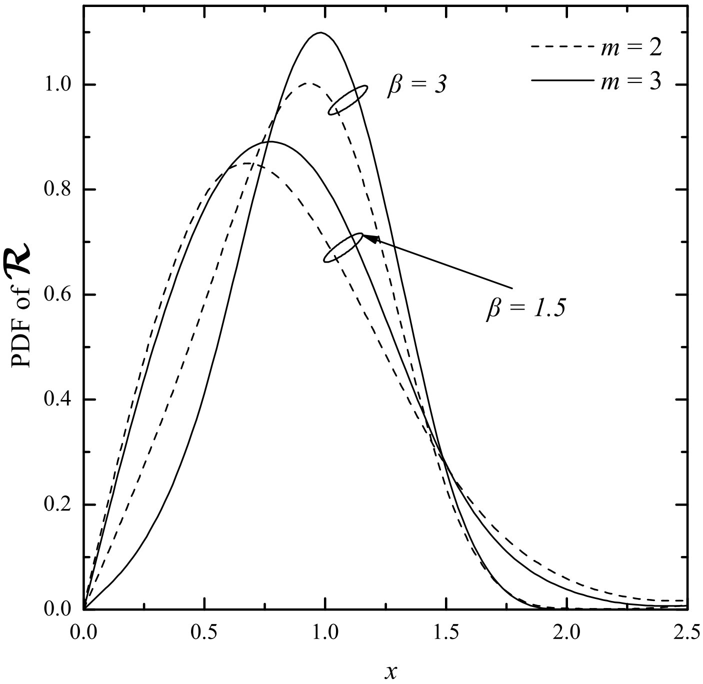

Fig. 1. The pdf of the product of two correlated GG rvs for $\beta = 1.5, 3$ and $m = 2, 3$.

2.2 Bound for the distribution of the sum

Let $\mathcal{K}$ be a rv defined as

$$\mathcal{K} \stackrel{\triangle}{=} X_1 + X_2. \tag{6}$$

Based on an inequality between the arithmetic, $\mathcal{A}$, and geometric, $\mathcal{G}$, mean [Abramowitz and Stegun, 1972, eq. (3.2.1)], $\mathcal{A} \geq \mathcal{G}$, with $\mathcal{A} \stackrel{\triangle}{=} \frac{1}{2}\,(X_1 + X_2)$ and $\mathcal{G} \stackrel{\triangle}{=} (X_1\, X_2)^{1/2}$, respectively, $\mathcal{K}$ can be lower bounded as

$$\mathcal{K} \geq 2\,\sqrt{\mathcal{R}}. \tag{7}$$

Using (3), (5), and (7), it can be easily seen that the cdf of $\mathcal{K}$ can be upper bounded as $F_{\mathcal{K}}(x) \leq F_{\mathcal{R}}[(x/2)^2]$. Hence, a union upper bound for the cdf of $\mathcal{K}$ yields as

$$\begin{aligned} F_{\mathcal{K}}(x) \leq & \sum_{k=0}^{\infty} \frac{2\,\rho^k\, m^{2(k+m)}\,(x/2)^{4(k+m)\beta}}{(1-\rho)^{2k+m}\Gamma(m)\Gamma(m+k)k!(k+m)^2\lambda_1^{k+m}} \\ & \times \left\{(k+m)\,{}_1F_2\left[1; k+m+1, k+m; \xi^2\,(x/2)^{4\beta}\right] K_0\left[2\,\xi\,(x/2)^{2\beta}\right] \right. \\ & \left. +\xi\,(x/2)^{2\beta}\, K_1\left[2\,\xi\,(x/2)^{2\beta}\right] {}_1F_2\left[1; k+m+1, k+m+1; \xi^2\,(x/2)^{4\beta}\right]\right\}. \end{aligned} \tag{8}$$

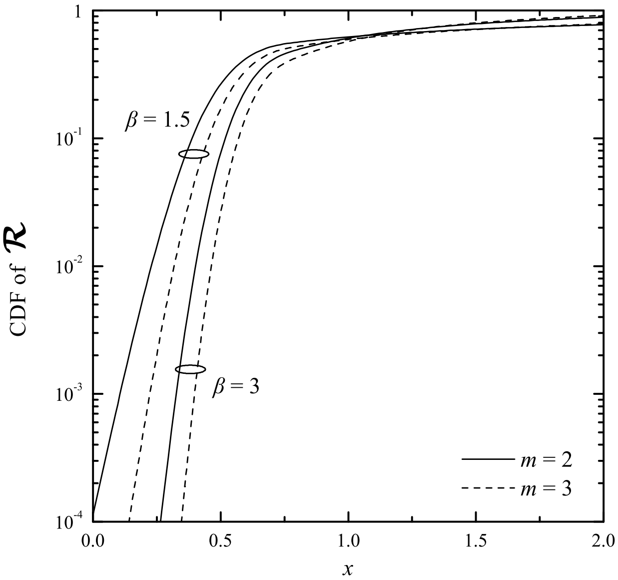

Fig. 2. The cdf of the product of two correlated GG rvs for $\beta = 1.5, 3$ and $m = 2, 3$.

In Figs. 1 and 2, the pdf and the cdf of $\mathcal{R}$ are plotted as a function of x, respectively. These results have been obtained for several values of the shaping parameters m, β and for $\rho = 0.5$, $\lambda_1 = 0.99$. Also, the effects of β and m on the cdf of $\mathcal{R}$ are illustrated in Fig. 2. Moreover, in Fig. 3 a bound for the cdf of $\mathcal{K}$ is plotted as a function of x for the same values of m, β, ρ, and λ_1. In the same figure, computer simulation results are also included, for the exact cdf of $\mathcal{K}$ for the same set of parameters as in Figs. 1 and 2. From the comparison, it can be easily verified the tightness of the proposed bounds. The higher the m and β are, the the tighter the bounds are.

3 Distribution of the ratio

Let $\mathcal{D}$ be a rv defined as

$$\mathcal{D} \triangleq \frac{X_1}{X_2}. \tag{9}$$

By using (1) and (9) with [Papoulis, 2001, eq. (6-43)]), and after a lot of algebraic manipulations, the pdf of $\mathcal{D}$ can be expressed in terms of standard functions as

$$f_{\mathcal{D}}(x) = \frac{2\beta\, 2^{2\,m-1}}{\sqrt{\pi}\,(1-\rho)^{-m}} \frac{\Gamma(m+1/2)}{\Gamma(m)} \frac{x^{2\beta\,m-1}\,\left(x^{2\beta}+\lambda_2\right)}{\left[\left(x^{2\beta}+\lambda_2\right)^2 - 4\,\rho\,\lambda_2\,x^{2\beta}\right]^{m+1/2}} \tag{10}$$

with $\lambda_2 = a_1/a_2$.

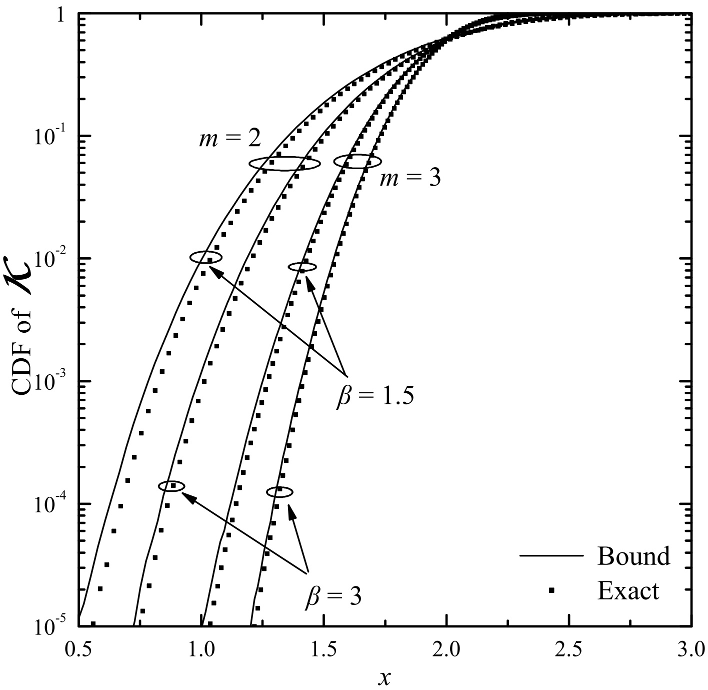

Fig. 3. The cdf of the sum of two correlated GG rvs for $\beta = 1.5, 3$ and $m = 2, 3$.

In Fig. 4 a few curves for the pdf of $\mathcal{D}$ are plotted for the same parameters as those in Figs. 1 and 2, with $\lambda_2 = 1.222$.

Starting from (10), the derivation of the cdf of $\mathcal{D}$ needs the calculation of the integral $F_{\mathcal{D}}(y) = \int_0^y f_{\mathcal{D}}(x)\,\mathrm{d}x$, where two integrals of the form

$$\int_0^{y^{2\beta}} \frac{x^m}{\sqrt{R^{2m+1}(x)}}\,\mathrm{d}x$$

need to be solved, with $R(x) > 0$ being a quadratic polynomial having a negative discriminant. The above type of integrals can be recursively solved for integer values of m using [Gradshteyn and Ryzhik, 2000, eqs. (2.263/1) and (2.263/4)].

4 Conclusions

The pdf and the cdf of the product of two correlated GG rvs, originated by Gaussian rvs, were derived. Based on an inequality between the arithmetic and geometric means, a union upper bound for the cdf of the sum of two correlated GG rvs was extracted. Moreover, the pdf of the ratio of two correlated GG rvs was obtained in closed form.

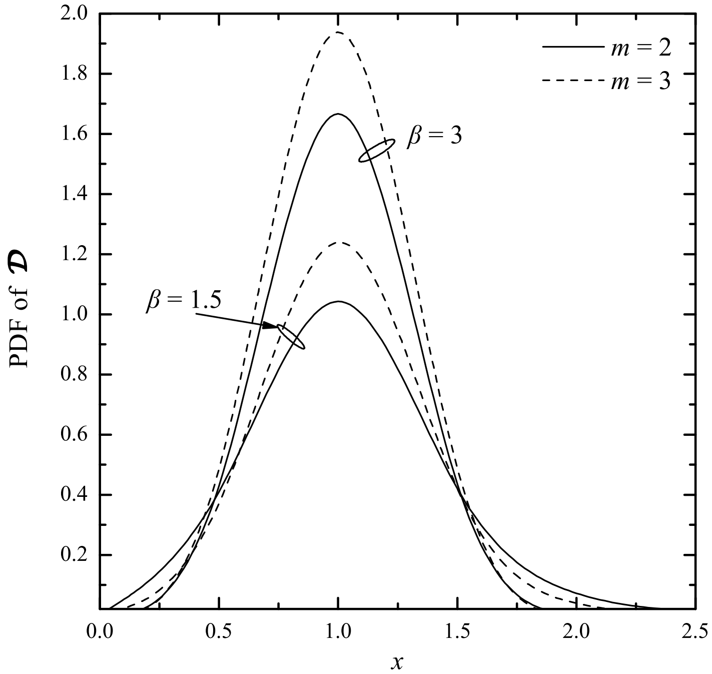

Fig. 4. The pdf of the ratio of two correlated rvs for $\beta = 1.5$, 3 and $m = 2, 3$.

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