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ON THE SUM OF SQUARED CORRELATED RAYLEIGH VARIATES AND APPLICATIONS TO MAXIMAL-RATIO DIVERSITY

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Abstract

An infinite series representation for the moment generating function of the sum of squared arbitrarily correlated Rayleigh random variables is presented. Based on the derived formula, corresponding analytical expressions for the probability density and cumulative distribution functions are extracted. As an application for the aforementioned sum, exact analytical expressions for the outage and the average error probability, as well as, the channel average spectral efficiency of multibranch maximal-ratio diversity receivers operating over identically distributed and arbitrarily correlated Rayleigh fading channels are obtained. Our analysis is verified by comparing numerically evaluated results with extensive computer simulation ones.

I Introduction

Diversity reception can significantly improve the performance of contemporary digital wireless communications systems in the presence of multipath fading and cochannel interference (CCI) [1]. Many of the performance analysis problems, that arise in the study of diversity combining receivers, require determination of the statistics of the sum of the squared envelopes of faded signals. Most known applications, where such sums can be useful, are maximal-ratio combining (MRC) and postdetection equal-gain combining (EGC), as well as, cellular systems with CCI (see [2–13] and references therein).

A very general approach for the distribution of the sum of gamma random variables (RVs) has been presented by Moshopoulos [2], where an infinite series representation for the probability density function (PDF) of the sum of independent non-identical gamma RVs has been proposed. Alouini *et al.* [3] have extended [2] for the case of arbitrarily correlated gamma RVs, and studied the performance of MRC and postdetection EGC receivers, as well as, receivers in the presence of CCI. Also, in [4–12], by using the characteristic function or the moment generating function (MGF), performance analytical formulas in the form of either infinite sums or higher order statistics of the fading parameter have been derived. Recently, in [13], closed-form expressions for the PDF and cumulative distribution function (CDF) of the sum of non-identical squared correlated Nakagami-m RVs, with integer-order fading parameters, have been presented. By following the PDF-based approach, analytical expressions for the performance of multibranch MRC and postdetection squared-law combining receivers have been studied.

In this paper, we provide a new closed-form expression for the MGF of the sum of squared arbitrarily correlated Rayleigh RVs. By using the Householder tridiagonalization method and a standard RVs transformation, a simple analytical expression for the MGF of the aforementioned sum is extracted which is used for the derivation of exact analytical expressions for the PDF and CDF of the sum of squared arbitrarily correlated Rayleigh RVs. Based on the proposed mathematical analysis, we provide a significant theoretical tool that can be efficiently used for the performance analysis of multibranch MRC receivers operating over identically distributed (id) and arbitrarily correlated Rayleigh fading channels. More specifically, simple exact analytical expressions for the outage probability (OP) and average symbol error probability (ASEP), as well as, the channel average spectral efficiency (SE) are obtained. Our analysis is also verified by extensive computer simulations.

II Statistics For the Sum of Squared Correlated Rayleigh RVs

Let $\mathbf{Y}_1 = [Y_{1,1}\, Y_{1,2} \cdots Y_{1,L}]^T$, $\mathbf{Y}_2 = [Y_{2,1}\, Y_{2,2} \cdots Y_{2,L}]^T$ be two L-dimensional real column vectors (T denotes the transpose), which are independent and id zero mean $\mathbb{E}\langle Y_{k,l}\rangle = 0$ with variance $\mathbb{E}\langle Y_{k,l}^2\rangle = \sigma^2$ ($k = 1,2$, $l = 1,2,\ldots,L$, and $\mathbb{E}\langle\cdot\rangle$ denotes expectation) Gaussian RVs having a symmetric and positive definite correlation matrix $\mathbf{\Sigma_G} \in \Re^{L\times L}$. Also, let $R_l = \|\mathbf{X}_l\| = \sqrt{Y_{1,l}^2 + Y_{2,l}^2}$ be the Euclidean norm of the two dimensional column vector $\mathbf{X}_l = [Y_{1,l}\, Y_{2,l}]^\dagger$ composed of the lth components of $\mathbf{Y}_k$'s. Clearly, R_l's are correlated Rayleigh RVs with marginal PDFs described by

$$f_{R_l}(r) = \frac{2r}{\Omega}\exp\left(-\frac{r^2}{\Omega}\right) \tag{1}$$

where $\Omega = 2\sigma^2 = \mathbb{E}\langle R_l^2\rangle$. Their power correlation matrix $\mathbf{\Sigma} \in \Re^{L\times L}$ is given by $\mathbf{\Sigma}_{i,j} \equiv 1$ for $i = j$ and $\mathbf{\Sigma}_{i,j} = \mathbf{\Sigma}_{j,i} \equiv \rho_{i,j}$ for $i \neq j$, with $0 \leq \rho_{i,j} < 1$ ($i,j = 1,2,\ldots,L$) being the power correlation coefficient (i.e., between R_i^2 and R_j^2) [1, eq. (9.195)]. It can be easily proved that the correlation matrix of the underlying Gaussian processes, $\mathbf{\Sigma_G}$, is related to the power correlation matrix, $\mathbf{\Sigma}$, as $\mathbf{\Sigma_G} = \sqrt{\mathbf{\Sigma}}$ ($\sqrt{\mathbf{\Sigma}}$ stands for a matrix with elements the square root ones of $\mathbf{\Sigma}$).

A Moment Generating Function

Let $Z_L = \sum_{l=1}^{L} R_l^2$ be the sum of squared arbitrarily correlated Rayleigh RVs. By applying a similarity transformation [15] to the inverse of the Gaussian correlation matrix, $\mathbf{W} = \mathbf{\Sigma_G}^{-1}$,

$$\mathbf{W}' = \mathbf{Q}^\dagger\,\mathbf{W}\,\mathbf{Q} \tag{2}$$

$\mathbf{W}'$ becomes real, symmetric, and tridiagonal, where $\mathbf{Q}$ is an orthogonal matrix given by a product of $L-2$ properly chosen Householder matrices [16].

Theorem 1 (Moment generating function) *The MGF of Z_L is given by*

$$\mathcal{M}_{Z_L}(s) = \frac{|\mathbf{W}|}{\Omega^L} \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \left[\prod_{i=1}^{L-1} \left(\frac{p_{i,i+1}^{k_i}}{\Omega^{k_i}\, k_i!} \right)^2 \right] \times \prod_{l=1}^{L} (b_l - 1)!\, (s + A_l)^{-b_l} \tag{3}$$

where $|\mathbf{W}|$ stands for the determinant of $\mathbf{W}$, $p_{i,j} \in \Re$ are the elements of $\mathbf{W}'$, $A_l = p_{l,l}/\Omega$, $b_1 = k_1 + 1$, $b_L = k_L + 1$, and $b_j = k_{j-1} + k_j + 1\, \forall j = 2, 3, \ldots, L-1$.

Proof: See [14]. ■

B Probability Density Function

The PDF of Z_L can be extracted as

$$f_{Z_L}(z) = \mathbb{L}^{-1}\{\mathcal{M}_{Z_L}(s); z\} \tag{4}$$

with $\mathbb{L}^{-1}\{\cdot;\cdot\}$ denoting the inverse Laplace transform. In order to evaluate (4), the integration theory of rational functions [17, Section 2.102] can be applied. Moreover, inverse Laplace transformations of the form $\mathbb{L}^{-1}\{(s + A_l)^{-q}; t\}$, with q integer, need to be performed, which using [17, Section 17.1] can be solved as

$$\mathbb{L}^{-1}\left\{(s + A_l)^{-q}; t\right\} = \frac{t^{q-1}}{(q-1)!} \exp(-A_l\, t). \tag{5}$$

Hence, assuming distinct values for A_l's[1] and after a lot of algebraic manipulations, a useful expression for the PDF of Z_L can be obtained as

$$f_{Z_L}(z) = \frac{|\mathbf{W}|}{\Omega^L} \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \left[\prod_{i=1}^{L-1} \left(\frac{p_{i,i+1}^{k_i}}{\Omega^{k_i}\, k_i!} \right)^2 \right] \times \left[\prod_{l=1}^{L} (b_l - 1)! \right] \sum_{p=1}^{L} \sum_{q=1}^{b_p} \frac{B_{p,q}}{(q-1)!} z^{q-1} \exp(-A_p\, z) \tag{6}$$

with

$$B_{p,q} = \frac{\Psi_p(s)^{(n_p - q)}|_{s=-A_p}}{(b_p - q)!} \tag{7}$$

and

$$\Psi_p(s) = (s + A_p)^{b_p} \prod_{l=1}^{L} (s + A_l)^{-b_l}. \tag{8}$$

Note that (6) agrees with [13, Theorem 2] for $m_l = 1\, \forall l$.

[1]Note that since $p_{l,l}$'s are distinct, A_l's are, generally, distinct.

C Cumulative Distribution Function

The CDF of Z_L can be obtained as

$$F_{Z_L}(z) = \mathbb{L}^{-1}\left\{ \frac{\mathcal{M}_{Z_L}(s)}{s}; z \right\} \tag{9}$$

where transformations of the form $\mathbb{L}^{-1}[(s + A_l)^{-q}/s; t]$ (q integer) can be solved, using [17, Section 17.1], as

$$\mathbb{L}^{-1}\left[\frac{(s + A_l)^{-q}}{s}; t \right] = \frac{\gamma(q, A_l\, t)}{(q-1)!\, A_l^q} \tag{10}$$

with $\gamma(\cdot,\cdot)$ being the lower incomplete gamma function, which can be further simplified to standard functions [17, eq. (8.352/2)]. Hence, assuming distinct values for A_l's, an analytical expression for the CDF of Z_L can be derived as

$$F_{Z_L}(z) = \frac{|\mathbf{W}|}{\Omega^L} \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \left[\prod_{i=1}^{L-1} \left(\frac{p_{i,i+1}^{k_i}}{\Omega^{k_i}\, k_i!} \right)^2 \right] \times \left[\prod_{l=1}^{L} (b_l - 1)! \right] \sum_{p=1}^{L} \sum_{q=1}^{b_p} \frac{B_{p,q}}{A_p^q} \times \left[1 - \exp(-A_p\, z) \sum_{n=0}^{q-1} \frac{1}{n!} (A_n\, z)^n \right]. \tag{11}$$

It is noted that (11) agrees with [13, Lemma 1] for $m_l = 1\, \forall l$.

III PERFORMANCE ANALYSIS OF MULTIBRANCH MRC RECEIVERS

We consider an L-branch diversity receiver operating over id and arbitrarily correlated Rayleigh fading channels. Let a signal's transmission over the lth flat Rayleigh fading channel ($l = 1, 2, \ldots, L$) corrupted by additive white Gaussian noise (AWGN), with E_s being the transmitted symbols' energy and N_0 the single-sided noise power spectral density of the AWGN. The instantaneous signal-to-noise ratio (SNR) per symbol of the lth diversity channel can be expressed by $\gamma_l = R_l^2\, E_s/N_0$, with its corresponding average value being $\overline{\gamma}_l = \mathbb{E}\langle R_l^2 \rangle\, E_s/N_0 = \Omega\, E_s/N_0 = \overline{\gamma}_s\ \forall\, l$.

The derived expressions of Section II are helpful in the study of several performance criteria of MRC receivers such as the OP, average channel SE, and ASEP.

A Outage Probability (OP)

The OP, P_{out}, in noise limited systems is defined as the probability that the instantaneous MRC output SNR falls below a given outage threshold, γ_{th}. This probability can be easily obtained, using (11), as

$$P_{\text{out}}\left(\gamma_{\text{th}}\right) = \frac{|\mathbf{W}|}{\left(\overline{\gamma}_s\right)^L} \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \left[\prod_{i=1}^{L-1} \frac{\left(p_{i,i+1}/\overline{\gamma}_s\right)^{2k_i}}{\left(k_i!\right)^2}\right] \times \left[\prod_{l=1}^{L} (b_l-1)!\right] \sum_{p=1}^{L}\sum_{q=1}^{b_p} \frac{B_{p,q}}{A_p^q} \times \left[1-\exp\left(-A_p\,\gamma_{\text{th}}\right)\sum_{n=0}^{q-1}\frac{1}{n!}\left(A_n\,\gamma_{\text{th}}\right)^n\right]. \tag{12}$$

B Average Spectral Efficiency (SE)

As it is well known, the Shannon channel capacity provides an upper bound of maximum transmission rate in a given Gaussian environment. The average SE in Shannon's sense, defined as the normalized (by the transmitted signal's bandwidth) average channel capacity, is given by

$$\overline{S}_e = \mathbb{E}\left\langle \log_2\left(1+\frac{E_s}{N_0}\,Z_l\right)\right\rangle. \tag{13}$$

Using (6) in the above equation, an integral of the form $\int_0^\infty \gamma^{q-1}\,\ln(1+\gamma)\,\exp\left(-A_p\,\gamma\right)\,\mathrm{d}\gamma$, appears. This type of integral has been solved in [18], by expressing the logarithmic and exponential integrands for arbitrary values of q as Meijer's G-functions [19]. Thus, the average channel capacity over arbitrarily correlated Nakagami-m fading can be obtained as

$$\overline{S}_e = \frac{|\mathbf{W}|}{\ln(2)\,\left(\overline{\gamma}_s\right)^L} \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \left[\prod_{i=1}^{L-1} \frac{\left(p_{i,i+1}/\overline{\gamma}_s\right)^{2k_i}}{\left(k_i!\right)^2}\right] \times \left[\prod_{l=1}^{L} (b_l-1)!\right] \sum_{p=1}^{L}\sum_{q=1}^{b_p} \frac{B_{p,q}}{(q-1)!}\, G_{2,3}^{\,3,1}\left[A_p\left|^{-q,\,1-q}_{0,\,-q,\,-q}\right.\right]. \tag{14}$$

C Average Symbol Error Probability (ASEP)

Based on (3), the ASEP, $\overline{P}_{se}$, at the output of an L-branch MRC receiver, for non-coherent binary frequency shift keying (NBFSK) and differential binary phase shift keying (DBPSK) modulation schemes can be directly calculated. For other schemes, including binary phase shift keying (BPSK), quadrature phase shift keying (QPSK),M-ary phase shift keying (M-PSK), quadrature amplitude modulation (M-QAM), amplitude modulation (M-AM), and differential phase shift keying (M-DPSK), single integrals with finite limits and integrands composed of elementary (exponential and trigonometric) functions, have to be readily evaluated via numerical integration [1].

IV Numerical and Computer Simulation Results

The numerical evaluation of several expressions in Section III requires the summation of an infinite number of terms. As indicative examples, Tables 1 and 2 summarize the number of terms needed for an MRC receivers so as the expressions for the OP using (12) and the ABEP using (3) to converge after the truncation of the infinite series, respectively. The findings are not very different, concerning the convergence, if (14) was used. Note that in Table 2, as well as, in the examples for the error performance that follow, when the modulation order $M > 2$, Gray encoding is assumed, resulting to $\overline{P}_{be} = \overline{P}_{se}/\log_2(M)$. A linearly arbitrary correlation model with $L = 3$ [20, p. 886] has been considered in Table 1, while in Table 2, a linearly arbitrary model with $L = 5$, in which the correlation matrix is given by [8, eq. (40)], has been assumed. As Table 1 indicates, the number of required terms depends strongly on the normalized outage threshold, $\gamma_{\text{th}}/\overline{\gamma}_s$. As $\gamma_{\text{th}}/\overline{\gamma}_s$ decreases, less terms are required to be summed. Moreover, for a fixed $\gamma_{\text{th}}/\overline{\gamma}_s$, an increase on m results to an increase on the required number of terms that are essential to be summed in order the OP to converge. Similar conclusions for the convergence of the ABEP for DBPSK can be also extracted from Table 2. An increase on the average SNR per bit, $\overline{\gamma}_b$, results to a decrease of the required number of terms, and for a fixed $\overline{\gamma}_b$, the required number of terms for convergence increases with increasing m. It is interesting to be mentioned that additional convergence experiments were conducted for the OP and the ABEP, and the following findings where obtained. *i*) The convergence rate does not depend on the diversity order and *ii*) an increase on the correlation coefficients results to an increase of the required number of terms needed for convergence.

Table 1: Number of required terms for convergence to the sixth significant digit of the OP of MRC with a linearly arbitrary model ($L = 3$).

$\gamma_{\text{th}}/\overline{\gamma}_s$ (dB)	Terms
5	14
0	11
-5	7
-10	5
-15	2
-20	1

Table 2: Number of required terms for convergence to the sixth significant digit of the ABEP of DBPSK of MRC with a linearly arbitrary model ($L = 5$).

$\overline{\gamma}_b$ (dB)	Terms
-5	18
0	13
5	9
10	6
15	3

Based on (12), Fig. 1 demonstrates the numerically evaluated results for P_{out} as a function of the $\gamma_{\text{th}}/\overline{\gamma}_s$, for multibranch MRC receivers, with a linearly arbitrary correlation matrix for $L = 3$ given in [20, p. 886], an arbitrary correlation matrix for $L = 4$ given by [14, eq. (13)], a linearly arbitrary correlation matrix for $L = 5$ given in [8, eq. (40)], and an arbitrary correlation matrix for $L = 6$ given by [14, eq. (30)]. In this figure and for comparison purposes, P_{out} curves for $L = 1$ and $L = 2$ with $\rho_{1,2} = 0.3$ are also included. As expected, P_{out}

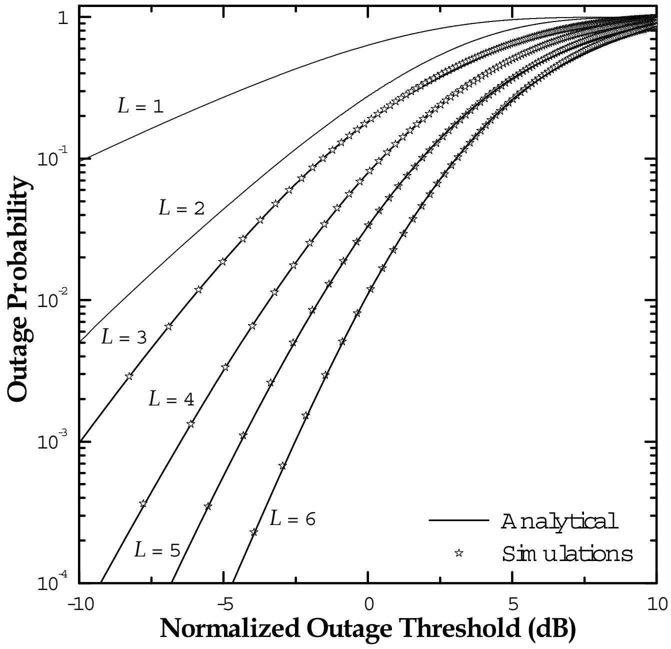

Figure 1: Outage probability of a multibranch MRC receiver as function of the normalized outage threshold for several correlation models.

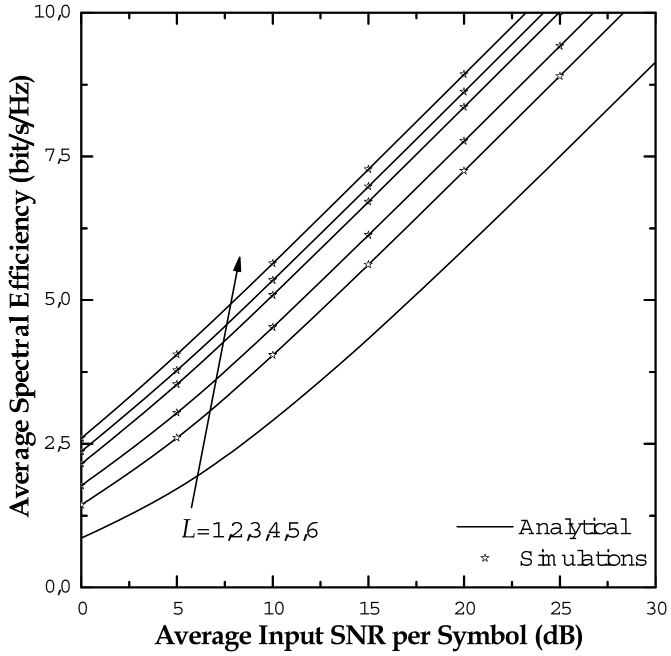

Figure 2: Average spectral efficiency versus average input SNR per symbol for several correlation models.

significantly improves as L increases. It is obvious that the numerically evaluated curves for P_{out} perfectly match with their corresponding simulation results for different values of L.

In Fig. 2, the normalized average SE, $\overline{S}_e$, is plotted as a function of the average SNR per symbol for multibranch MRC receivers, with the same as above correlation matrices

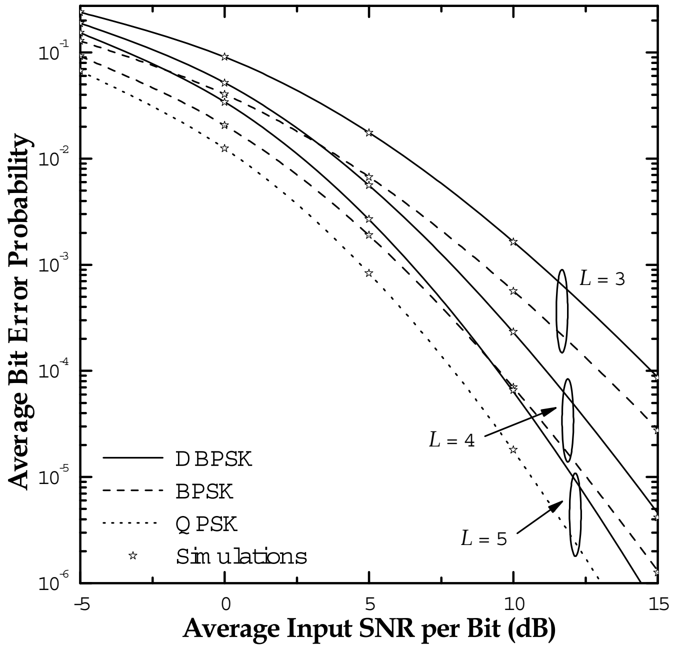

Figure 3: Average bit error probability of DBPSK, BPSK, and QPSK for multibranch MRC receiver as a function of the average input SNR per bit for several correlation models.

for $L = 3, 5, 6$ and for a constant correlattion matrix for $L = 4$ given by [20, p. 888]. In this figure and for comparison purposes, $\overline{S}_e$ curves for $L = 1$ and $L = 2$ with $\rho_{1,2} = 0.3$ are also included. As expected, $\overline{S}_e$ improves as L increases. Also, it is evident that the numerically evaluated curves for $\overline{S}_e$ perfectly match with their corresponding simulation results for different values of L.

Having numerically evaluated (3), in Fig. 3, the ABEP performance is plotted as a function of the $\overline{\gamma}_b = \overline{\gamma}_s / \log_2(M)$, for multibranch MRC receivers, several modulation schemes, with a linearly arbitrary correlation matrix for $L = 3$ given in [20, p. 886], a circular correlation matrix for $L = 4$ given in [20, p. 888], and a linearly arbitrary correlation matrix for $L = 5$ given in [8, eq. (40)]. As expected, the ABEP improves with an increase of $\overline{\gamma}_b$. Note also that as this figure indicates, the curves for numerically evaluated results for the ABEP coincide to the equivalent computer simulation ones for different values of L.

V Conclusions

In this paper, a new analytical expression for the MGF of the sum of squared arbitrarily correlated Rayleigh RVs was presented. Based on that useful formula, exact analytical expressions for the the PDF and CDF of the sum of squared arbitrarily correlated Rayleigh RVs were derived. More importantly, based on the proposed mathematical analysis, exact analytical expressions for the OP, the ASEP, as well as, the average SE of multibranch MRC receivers operating over id and arbitrarily correlated Rayleigh fading channels were obtained. Comparisons between numerically evaluated results and extensive computer simulation ones verified the validity of our approach.

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