Outage Analysis of Decode-and-Forward Relaying Over Nakagami-m Fading Channels

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Abstract—In this letter, closed-form outage probability expressions are presented for an *L*-relays dual-hop plus a direct link network, in which the decode-and-forward relaying protocol is employed. Our analysis significantly extends previous results on Rayleigh fading, considering a Nakagami-m fading environment with either equal or distinct second hops fading parameters to average power ratios. Various numerical examples illustrate the proposed analysis.

Index Terms—Decode-and-forward (DF), diversity, multihop networks, Nakagami-m, outage probability (OP), relays.

I. INTRODUCTION

MULTIHOP networks have several advantages over traditional communication systems in terms of deployment, connectivity, and capacity, while minimizing the need for fixed infrastructure [1]–[3]. A special class of multihop communication networks are the cooperative ones, where the destination terminal combines the signals received from the source as well as the relays terminals [4], [5]. The destination terminal can employ a variety of diversity techniques such as maximal-ratio, equal-gain, or selection diversity to benefit from the multiple signal replicas available [6]. Besides, there are various protocols that achieve the benefits of user cooperation. One of them is the decode-and-forward (DF) relaying protocol, which uses relays that encode and retransmit the signal towards the destination after demodulating and decoding the received signal from the source.

An important performance criterion which is useful for evaluating the transmission protocol under consideration is the outage probability (OP). In [7], a closed-form solution for the OP of DF relaying has been presented, assuming independent and identically distributed (id) relay channels among source, relays, and destination terminals. In addition, lower and asymptotic bounds have been proposed in [7] and [8], respectively, assuming independent but not id channels. In a recent contribution [9], Beaulieu and Hu have presented a closed-form expression for the OP in dissimilar Rayleigh

fading channels. Real wireless channels are more accurately modeled by more generic distributions than the Rayleigh one, such as the well-known Rice and Nakagami-m models [10].

Motivated by all of the above, in this letter, we extend [9] by analyzing the performance of DF dual-hop networks over Nakagami-m fading channels. More specifically, exact closed-form expressions are presented for the OP of DF relaying systems operating over either equal or distinct second hops Nakagami-m fading parameters to average signal-to-noise ratios (SNRs).

II. SYSTEM AND CHANNEL MODEL

We consider a dual-hop system with a source node, **S**, communicating with a destination node, **D**, via a direct link as well as through L relay nodes, $\mathbf{R}_\ell (\ell = 1, 2, \ldots, L)$. We assume that the receivers at the relays and destination have perfect channel state information (CSI) so that maximum-likelihood combining can be employed; also, no transmitter CSI is available at the source or the relays. In order to accomplish orthogonal transmission, a time-division channel allocation scheme with $L + 1$ time slots is used [7], [8]. During the first time slot, **S** broadcasts its signal to the set of L-relay nodes and also directly to **D**. We define the decoding set $\mathcal{C}$ as those relays with the ability to fully decode the source message. If the channel between the source and the relay node is strong enough to allow for successful decoding, the relay node is said to belong to $\mathcal{C}$. During the rest L time slots, the members of the decoding set first decode and then forward the source information to the destination node in a predetermined order.

The mutually independent channel envelopes among $\mathbf{S} \longrightarrow \mathbf{D}$, $\mathbf{S} \longrightarrow \mathbf{R}_\ell$, and $\mathbf{R}_\ell \longrightarrow \mathbf{D}$, represented by h_{SD}, h_{SR_ℓ}, and $h_{\text{R}_\ell D}$, respectively, are modeled as Nakagami-m random variables (RVs), while additive white Gaussian noise with N_0 single side power spectral density is also assumed. The corresponding powers $X = h_{\text{SD}}^2$, $Z_\ell = h_{\text{SR}_\ell}^2$, and $Y_\ell = h_{\text{R}_\ell D}^2$ are gamma distributed RVs. Let us define the instantaneous SNR of the second hop of the ℓth link as $\upsilon_\ell = Y_\ell^2 E_s/N_0$, with E_s being the symbol energy transmitted by **S**. This SNR also follows the gamma distribution with probability density function (PDF) given by[1]

$$f_{\upsilon_\ell}(\gamma) = \frac{B_\ell^{m_\ell}}{\Gamma(m_\ell)} \gamma^{m_\ell - 1} \exp(-\gamma B_\ell) \qquad (1)$$

where $m_\ell \geq 1/2$ denotes the Nakagami-m fading parameter, $\Gamma(\cdot)$ is the gamma function [11, (8.310/1)], and $B_\ell = m_\ell/\overline{v}_\ell$, with $\overline{v}_\ell = \mathbb{E}\langle Y_\ell^2 \rangle E_s/N_0$ ($\mathbb{E}\langle\cdot\rangle$ denotes expectation) being the corresponding average SNR. In our research, we assume that

[1]The parameters set of a gamma distributed RV of the form of (1) is denoted as $(m_\ell, \overline{v}_\ell)$.

B_ℓ's are either equal, i.e., $B_\ell = B\,\forall \ell$, or distinct, i.e., $B_\ell \neq B_k\,\forall \ell \neq k$. Also, the cumulative distribution function (CDF) of υ_ℓ is $F_{\upsilon_\ell}(\gamma) = 1 - \Gamma(m_\ell, B_\ell\gamma)/\Gamma(m_\ell)$, where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function [11, (8.350/2)]. We also define the instantaneous SNR of the direct link and the first hop of the ℓth link as $\sigma = X^2\, E_s/N_0$ and $\zeta_\ell = Z_\ell^2\, E_s/N_0$, which similarly to υ_ℓ are gamma distributed with parameters $(m_0, \overline{\sigma})$ and $(n_\ell, \overline{\zeta}_\ell)$, respectively.

III. Outage Probability of DF Relaying

The mutual information between the source and ℓth relay node is $I_\ell = \log[1 + (\mathcal{S}/\mathcal{N})\, Z_\ell]/(L+1)$, where $(\mathcal{S}/\mathcal{N})$ is the transmit SNR. Thus, if I_ℓ is higher than a fixed spectral efficiency, $\mathcal{R}$, $\mathbf{R}_\ell$ is able to successfully decode the transmitted signal from $\mathbf{S}$ and belongs to the decoding set $\mathcal{C}$. The end-to-end mutual information, when DF relaying is used, can then be expressed as $I_{\mathrm{DF}} = \log[1 + (\mathcal{S}/\mathcal{N})(X + \sum_{c\in\mathcal{C}} X_c)]/(L+1)$. By the total probability law, the end-to-end OP is given by

$$P_{\mathrm{out}} = \Pr[I_{\mathrm{DF}} < \mathcal{R}] = \sum_c \Pr[I_{\mathrm{DF}} < \mathcal{R}\,|\,\mathcal{C}]\Pr[\mathcal{C}]. \quad (2)$$

Similarly to [9], we view the wireless L-relay plus the direct channel as effectively shaping $L+1$ paths between the source and the destination. Let the zeroth path represent the $\mathbf{S} \longrightarrow \mathbf{D}$ direct link and the ℓth path represent the $\mathbf{S} \longrightarrow \mathbf{R}_\ell \longrightarrow \mathbf{D}$ cascaded link. Also, let γ_ℓ denote the received SNR of both $\mathbf{S} \longrightarrow \mathbf{R}_\ell$ and $\mathbf{R}_\ell \longrightarrow \mathbf{D}$ links. Then, γ_ℓ has PDF (link active: la, link not active: lna)

$$f_{\gamma_\ell}(\gamma) = f_{\gamma_\ell\,|\,\mathrm{lna}}(\gamma)\Pr[\mathrm{lna}] + f_{\gamma_\ell\,|\,\mathrm{la}}(\gamma)\Pr[\mathrm{la}] \quad (3)$$

with $f_{\gamma_\ell\,|\,\mathrm{la}}(\gamma)$ and $f_{\gamma_\ell\,|\,\mathrm{lna}}(\gamma)$ being the conditional PDFs, and $\Pr[\mathrm{la}]$ and $\Pr[\mathrm{lna}]$ the associated probabilities for the ℓth link to be active and not active, respectively. When the ℓth link is not active, $f_{\gamma_\ell\,|\,\mathrm{lna}}(\gamma) = \delta(\gamma)$, with $\delta(\cdot)$ being the Dirac function, and the associated probability is $A_\ell = \Pr\{\zeta_\ell < [2^{(L+1)\mathcal{R}} - 1]/(\mathcal{S}/\mathcal{N})\}$, which can be expressed as

$$A_\ell = F_{\zeta_\ell}\left\{\left[2^{(L+1)\mathcal{R}} - 1\right]/(\mathcal{S}/\mathcal{N})\right\} \quad (4)$$

with $F_{\zeta_\ell}(\cdot)$ denoting the CDF of ζ_ℓ. The above equation represents the probability that the ℓth relay node does not belong to $\mathcal{C}$. Note that since the direct $\mathbf{S} \longrightarrow \mathbf{D}$ path is not linked via a relay, $A_0 = 0$. The probability that the ℓth link is active is $1 - A_\ell$, and the associated conditional PDF is $f_{\gamma_\ell\,|\,\mathrm{la}}(\gamma) = f_{\upsilon_\ell}(\gamma)$. Hence, substituting (1) in (3) yields[2]

$$f_{\gamma_\ell}(\gamma) = A_\ell\delta(\gamma) + (1 - A_\ell)B_\ell^{m_\ell}\frac{\gamma^{m_\ell - 1}}{\Gamma(m_\ell)}\exp(-\gamma B_\ell). \quad (5)$$

The above PDF represents the ℓth cascaded path from source to destination.

[2]Hereafter, $\ell = 0, 1, \ldots, L$.

The end-to-end OP can be written as

$$P_{\mathrm{out}} = \Pr\left(\sum_{\ell=0}^{L} \gamma_\ell < \gamma_{\mathrm{th}}\right) \quad (6)$$

with $\gamma_{\mathrm{th}} = [2^{(L+1)\,\mathcal{R}} - 1]/(\mathcal{S}/\mathcal{N})$ being the outage threshold. Equation (6) shows that the OP in this model is simply the CDF of the end-to-end SNR $\gamma_{\mathrm{end}} = \sum_{\ell=0}^{L} \gamma_\ell$ evaluated at γ_{th} (by design). In order to extract this CDF, we first obtain the moment-generating function (MGF) of γ_ℓ, i.e., $M_{\gamma_\ell}(s) = \mathbb{E}\langle\exp(-s\,\gamma_\ell)\rangle$. Using (5), this MGF can be obtained as

$$M_{\gamma_\ell}(s) = A_\ell + (1 - A_\ell)(1 + s/B_\ell)^{-m_\ell}. \quad (7)$$

Moreover, since γ_ℓ's are independent, the MGF of γ_{end} can be expressed as

$$M_{\gamma_{\mathrm{end}}}(s) = \prod_{\ell=0}^{L} M_{\gamma_\ell}(s) \quad (8)$$

while the end-to-end OP can be extracted as

$$P_{\mathrm{out}} = \mathbb{L}^{-1}\{M_{\gamma_{\mathrm{end}}}(s)/s; t\}|_{t=\gamma_{\mathrm{th}}} \quad (9)$$

with $\mathbb{L}^{-1}\{\cdot; \cdot\}$ denoting the inverse Laplace transformation.

A. Distinct B_ℓ's

We consider the case of distinct B_ℓ's, i.e., $B_\ell \neq B_k\,\forall \ell \neq k$, and integer-order fading parameters. By substituting (7) in (8), a product of $L+1$ terms $\prod_{k=0}^{L}(1 + B_k)$ is formed. This product can be expanded using the following useful formula that we have developed:

$$\prod_{k=0}^{L}(1 + B_k) = 1 + \sum_{k=0}^{L}\sum_{\lambda_0=0}^{L-k}\sum_{\lambda_1=\lambda_0+1}^{L-k+1} \cdots \sum_{\lambda_k=\lambda_{k-1}+1}^{L}\prod_{n=0}^{k} B_{\lambda_n}$$

helping us to simplify the integration that appears next using (9). Taking into account that the fading parameters are considered to be integers, the integration theory of rational functions [11, Section 2.102] can be applied. Moreover, inverse Laplace transformations of the form $\mathbb{L}^{-1}\{(s + B_\ell)^{-q}/s; t\}$, with q integer, need to be performed, which using [11, Section 17.1] can be solved as

$$\mathbb{L}^{-1}\{(s + B_\ell)^{-q}/s; t\} = \gamma(q, B_\ell t)/[\Gamma(q)\, B_\ell^q]$$

with $\gamma(q, x)$ being the lower incomplete gamma function, $\gamma(q, x) = \Gamma(q) - \Gamma(q, x)$, which can be further simplified to standard functions [11, (8.352/2)]. Hence, after a lot of algebraic manipulations, a closed-form expression for the end-to-end OP

of DF relaying with L-relay nodes over Nakagami-m fading with integer-order values for m_ℓ's can be obtained as

$$
\begin{aligned}
P_{\text{out}} &= \left(\prod_{i=0}^{L} A_i\right)\left\{1 + \sum_{k=0}^{L}\sum_{\lambda_0=0}^{L-k}\sum_{\lambda_1=\lambda_0+1}^{L-k+1}\cdots\right.\\
&\quad \sum_{\lambda_k=\lambda_{k-1}+1}^{L}\left(\prod_{n=0}^{k}\frac{1-A_{\lambda_n}}{A_{\lambda_n}}B_{\lambda_n}^{m_{\lambda_n}}\right)\\
&\quad \times\sum_{p=0}^{k}\sum_{q=1}^{m_{\lambda_p}}\frac{\xi_q}{B_{\lambda_p}^q}\left[1-\exp\left(-\gamma_{\text{th}}B_{\lambda_p}\right)\right.\\
&\quad \left.\left.\times\sum_{l=0}^{q-1}\frac{1}{l!}\left(\gamma_{\text{th}}B_{\lambda_p}\right)^l\right]\right\}
\end{aligned}
\tag{10}
$$

with $\xi_q = \Psi_p(s)^{(m_{\lambda_p}-q)}|_{s=-B_{\lambda_p}}/(m_{\lambda_p}-q)!$ and $\Psi_p(s) = (s+B_{\lambda_p})^{m_{\lambda_p}}\prod_{n=0}^{k}(B_{\lambda_n}+s)^{-m_{\lambda_n}}$. Note that for Rayleigh fading, i.e., $m_\ell = 1\forall\ell$, (10) agrees with [9, equation (10a)].

B. Equal B_ℓ's

For equal B_ℓ's, i.e., $B_\ell = B\forall\ell$ with arbitrary m_ℓ, following a similar procedure as that in Section III-A, a closed-form expression for the end-to-end OP can be obtained as

$$
\begin{aligned}
P_{\text{out}} &= \left(\prod_{i=0}^{L} A_i\right)\left[1 + \sum_{k=0}^{L}\sum_{\lambda_0=0}^{L-k}\sum_{\lambda_1=\lambda_0+1}^{L-k+1}\cdots\right.\\
&\quad \sum_{\lambda_k=\lambda_{k-1}+1}^{L}\left(\prod_{n=0}^{k}\frac{1-A_{\lambda_n}}{A_{\lambda_n}}\right)\\
&\quad \left.\times\gamma\left(\sum_{n=0}^{k}m_{\lambda_n},B\gamma_{\text{th}}\right)/\Gamma\left(\sum_{n=0}^{k}m_{\lambda_n}\right)\right].
\end{aligned}
\tag{11}
$$

Moreover, for integer values of m_ℓ, (11) simplifies to

$$
\begin{aligned}
P_{\text{out}} &= \left(\prod_{i=0}^{L} A_i\right)\left\{1 + \sum_{k=0}^{L}\sum_{\lambda_0=0}^{L-k}\sum_{\lambda_1=\lambda_0+1}^{L-k+1}\cdots\right.\\
&\quad \sum_{\lambda_k=\lambda_{k-1}+1}^{L}\left(\prod_{n=0}^{k}\frac{1-A_{\lambda_n}}{A_{\lambda_n}}\right)\\
&\quad \times\left[1-\exp(-\gamma_{\text{th}}B)\right.\\
&\quad \left.\left.\times\sum_{l=0}^{\sum_{n=0}^{k}m_{\lambda_n}-1}\frac{(\gamma_{\text{th}}B)^l}{l!}\right]\right\}.
\end{aligned}
\tag{12}
$$

IV. RESULTS AND DISCUSSION

Without loss of generality, we assume $\mathcal{R} = 1$ bit/s/Hz and equal fading parameters for both the first and the second hop of each one of the L links, i.e., $m_\ell = n_\ell$ and $\bar{v}_\ell = \bar{\zeta}_\ell\forall\ell$. Figs. 1 and 2 show the end-to-end OP with distinct and id Nakagami-m channel statistics, respectively, as a function of the transmit SNR, $\mathcal{S}/\mathcal{N}$. In Fig. 1, a few curves are plotted for $L =$
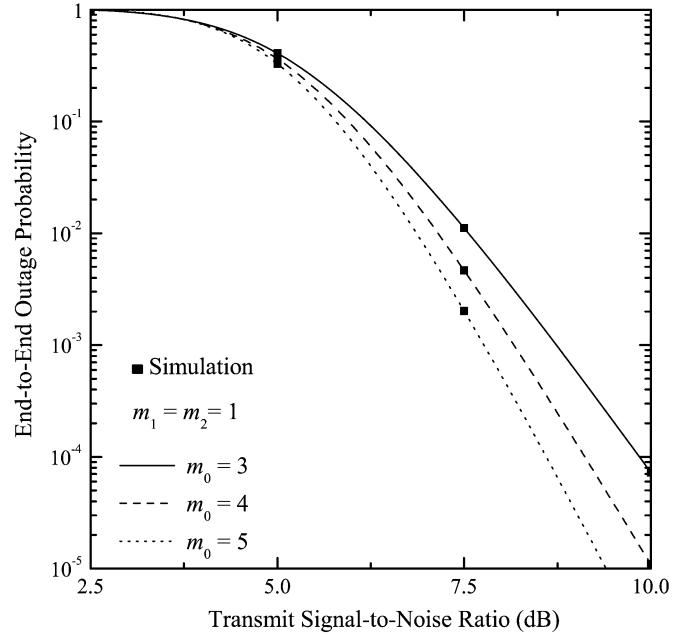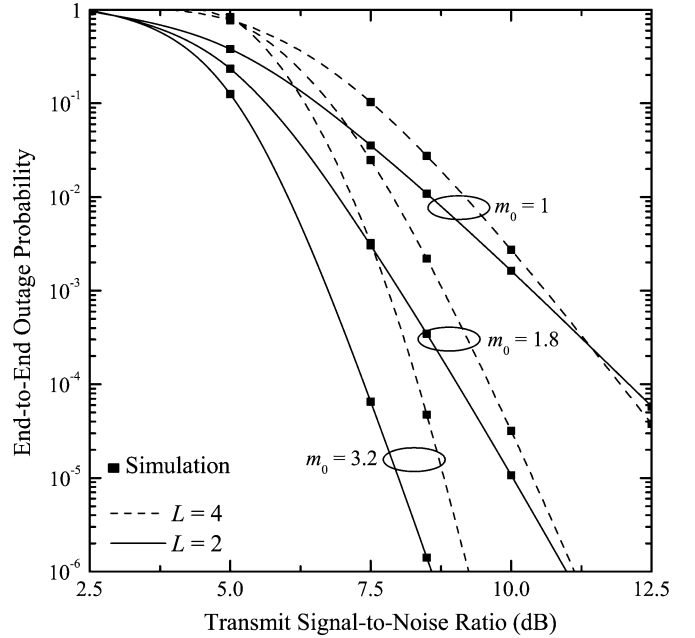

Fig. 1. End-to-end OP with two nodes and distinct B_ℓ's.

Fig. 2. End-to-end OP with two and four nodes and equal B_ℓ's.

2 nodes and $\bar{\zeta}_1 = 0.75\bar{\sigma}, \bar{\zeta}_2 = 0.5\bar{\sigma}$. Moreover, in Fig. 2, $L = 2$ and 4 nodes with $\bar{\zeta}_\ell = \bar{\sigma}$ and $m_\ell = m_0 = 1, 1.8, 3.2$ are considered. As expected, the OP decreases as $\mathcal{S}/\mathcal{N}$ increases. In both figures, the numerically evaluated results coincide with equivalent simulation ones, verifying the correctness of the presented analysis. Interestingly enough, the presence of many nodes does not necessarily improve the OP as also observed in [9]. For example, from Fig. 2, a certain $\mathcal{S}/\mathcal{N}$ threshold value can be found, where for lower values of that threshold, the OP is lower for two nodes compared to four, e.g., for $m_0 = 1$, this value is at 11.5 dB. As m_0 increases, this SNR value, for which the curves for $L = 2$ and 4 are crossed, moves towards lower OP values.

Hence, we may conclude that depending on the channels conditions, an optimal number of relaying nodes should be utilized, when the DF relaying protocol is employed.

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