Error Rate Performance of Multilevel Signals with Coherent Detection

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***Abstract*—In this paper, coherent detection for multilevel correlated signaling sets in additive white Gaussian noise is addressed. The contribution is twofold. Firstly, correlation structures that minimize the symbol error probability (SEP) of M-ary frequency-shift keying as well as of arbitrarily correlated signaling sets are investigated, while secondly, a general and analytical expression for the SEP is derived in the form of a single integral. The structure of the associated correlation matrix is generic and includes various known signaling sets as special cases. Specific correlation structures that minimize the SEP are also studied. Based on eigendecomposition or LU decomposition, generic methods for constructing a correlated signaling set for any correlation matrix under consideration are also provided.**

***Index Terms*—Coherent detection, correlated signals, correlation matrix, detection theory, digital modulation, frequency-shift keying (FSK), multivariate statistics, orthogonal signals.**

I. Introduction

THE rapidly increasing need for high data rates and high quality of services drives engineers to design systems that will provide optimum performance under a number of parameters and constraints existing in all layers. In the physical layer, there are many options that are available and must be carefully addressed. Some of them with major impact to system performance are the detection scheme, the modulation scheme and order, and the signaling set [1]–[6]. Among them, the choice of the detection scheme is critical. As it is well known, depending on whether or not the receiver is equipped with a phase recovery circuit, the detection scheme can be classified as coherent or noncoherent, with the former providing better performance at the expense of increased implementation complexity [7], [8]. Although many combinations of the various options are possible, some are more popular than others. A well-known combination met in practice is for multilevel frequency-shift keying (FSK) [9], where noncoherent detection with orthogonal signaling is commonly used [10]–[12]. However, neither noncoherent detection nor orthogonal signaling sets necessarily lead to optimum performance. For example, it is well known that in the case of binary FSK (BFSK) modulation, the best performance occurs for coherent reception with the correlation between the two available signals being equal to -0.2172 [1, Section 8.1.1.6]. It is therefore reasonable to seek for and study correlated signaling sets in conjunction with coherent receivers that minimize the probability of error.

This paper deals with coherent detection of correlated signals in additive white Gaussian noise (AWGN). Correlation structures that minimize the symbol error probability (SEP) of M-ary FSK as well as of arbitrarily correlated signaling sets are studied. Moreover, a general analytical expression for the SEP is derived with the associated correlation matrix being quite generic, including various known signaling sets as special cases. Based on the derived expression, specific correlation structures for signaling sets that minimize the SEP are studied. Moreover, based on eigendecomposition or LU decomposition, generic methods for constructing specific signaling sets for any correlation matrix under consideration are provided.

II. Coherent Detection of Multilevel Signals in AWGN

Let $r(t) \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) denote the received signal of modulation order M at the time instant t, where $0 \leq t \leq T_s$ and T_s is the symbol duration. We can express $r(t)$ as $r(t) = s(t) + n(t)$, where $n(t) \in \mathbb{R}$ is the AWGN with two-sided power spectral density of $N_0/2$, i.e., $\mathbb{E}\langle n(t)\rangle = 0$ and $\mathbb{E}\langle n(t_1)\, n(t_2)\rangle = (N_0/2)\,\delta(t_1 - t_2)\ \forall t_1, t_2$, with $\mathbb{E}\langle\cdot\rangle$ denoting the expectation operator, $\delta(\cdot)$ denoting the Dirac delta function, and $s(t) = \{s_1(t), s_2(t), \ldots, s_M(t)\} \in \mathbb{R}$ is one of the M available signals each having energy $E_i = \int_0^{T_s} s_i^2(t)\,\mathrm{d}t$ ($i = 1, 2, \ldots, M$). All M signals are assumed to have equal a priori probability of occurrence and are not necessarily orthogonal to each other, with the correlation coefficient between $s_i(t)$ and $s_j(t)$ ($i, j = 1, 2, \ldots, M$) being

$$\rho_{i,j} = \frac{1}{\sqrt{E_i\, E_j}} \int_0^{T_s} s_i(t)\, s_j(t)\,\mathrm{d}t\,, \tag{1}$$

with $-1 \leq \rho_{i,j} \leq 1$. Moreover, $s_i(t)$ can be expanded as a weighted sum of a set of orthonormal basis functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)$ of dimension $N \leq M$ as $s_i(t) = \sum_{j=1}^{N} s_{i,j}\,\varphi_j(t)$, with $\int_0^{T_s} \varphi_j(t)\,\varphi_k(t)\,\mathrm{d}t = \delta[j-k]$, where $\delta[\cdot]$ denotes the Kronecker delta function, and $s_{i,j} = \int_0^{T_s} s_i(t)\,\varphi_j(t)\,\mathrm{d}t$. When the ith signal $s_i(t)$ is transmitted, we can rewrite $r(t)$ in vector form as $\underline{r} = \underline{s}_i + \underline{n}$, where $\underline{s}_i = [s_{i,1}, s_{i,2}, \cdots, s_{i,N}]^T$, $(\cdot)^T$ denoting transpose, and $\underline{n}$

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is an N-dimensional zero-mean real Gaussian vector with covariance matrix $(N_0/2)\,\underline{I}_N$, $\underline{I}_N$ denoting the $N\times N$ identity matrix. Thus $\underline{n} \sim \mathcal{N}(\underline{0}_N, (N_0/2)\,\underline{I}_N)$, with $\underline{0}_N$ denoting the $N \times 1$ vector of zeros.

A. Coherent Detection

The coherently detected signal is given by the decision rule $\hat{\underline{s}}_i = \arg\min_{\underline{s}_j \in \{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_M\}} ||\underline{r} - \underline{s}_j||^2$ or

$$\hat{s}_i(t) = \underset{s_j(t)\in\{s_1(t),s_2(t),\ldots,s_M(t)\}}{\arg\max} \left[\int_0^{T_s} r(t)\,s_j(t)\mathrm{d}t - \frac{1}{2}\,E_j\right]. \tag{2}$$

When the ith signal is transmitted, a correct decision occurs under the condition $\hat{\underline{s}}_i = \underline{s}_i$. Based on (2) we define the random variable (RV) $L_j \triangleq \int_0^{T_s} r(t)\,s_j(t)\,\mathrm{d}t - E_j/2$ and under the hypothesis that the ith signal is transmitted among the M available, L_j can be expressed as

$$L_j(i) = \begin{cases} \rho_{i,j}\,\sqrt{E_i\,E_j} - \frac{1}{2}\,E_j + \int_0^{T_s} n(t)\,s_j(t)\,\mathrm{d}t\,, & i \neq j, \\ \frac{1}{2}\,E_i + \int_0^{T_s} n(t)\,s_i(t)\,\mathrm{d}t\,, & i = j. \end{cases} \tag{3}$$

It can be easily verified that $L_j(i) \sim \mathcal{N}(\rho_{i,j}\,\sqrt{E_i\,E_j} - E_j/2,\;\; E_j\,N_0/2)$. Let us define another vector $\underline{L}(i) \triangleq [L_1(i), L_2(i), \ldots, L_M(i)]^T$, with mean of the jth element $\mathbb{E}\langle L_j(i)\rangle = \rho_{i,j}\,\sqrt{E_i\,E_j} - E_j/2$, for $i \neq j$, and $\mathbb{E}\langle L_j(i)\rangle = E_i/2$, for $i = j$. Also, the corresponding covariance between the jth and the kth elements is $\mathrm{cov}[L_j(i), L_k(i)] = \rho_{j,k}\,\sqrt{E_j\,E_k}\,N_0/2$, for $j \neq k$, and $\mathrm{cov}\,[L_j(i), L_k(i)] = E_j\,N_0/2$, for $j = k$. We can now express the covariance matrix of $\underline{L}(i)$ as $\mathrm{cov}[\underline{L}(i)] = \sqrt{\underline{E}}\,\underline{R}_s\,\sqrt{\underline{E}}\,(N_0/2)$, where $\sqrt{\underline{E}}$ is an $M \times M$ diagonal matrix, the elements of which are the square roots of the symbol energies $E_1, E_2, \ldots, E_M$, and $\underline{R}_s$ is the symmetric *correlation matrix* of the signaling set, with elements $[\underline{R}_s]_{i,j} = \rho_{i,j}\ \forall i \neq j$ and $[\underline{R}_s]_{i,i} = 1\ \forall i$.

B. Symbol Error Probability

Under the hypothesis that the ith signal is transmitted, we find from (2) that the correct decision occurs when $L_i(i)$ is the largest among $L_1(i), L_2(i), \ldots, L_M(i)$. The probability of correct decision is therefore [1, eq. (5-2-17)], [3, eq. (60)]

$$P_{c_i} = \Pr\left[L_j(i) - L_i(i) < 0,\, j \neq i,\, j = 1, 2, \ldots, M\right]. \tag{4}$$

The analytical expression for P_{c_i} is given by [2, eq. (4.80)], while the SEP of coherently detected, equienergy, and arbitrarily correlated signals can be further obtained as

$$P_{es} = 1 - \frac{1}{M}\sum_{i=1}^{M} P_{c_i}\,. \tag{5}$$

C. Minimum Symbol Error Probability Analysis

Our intention is to find specific forms for $\underline{R}_s$ that minimize, or in general, lower P_{es}. A standard way to do this is by using an appropriate multidimensional minimization method, such as the steepest descent. However, the known form of P_{es} is complicated and it seems that none of the known methods can be helpful. Therefore, we have developed a computer simulation software and we focus on arbitrarily correlated M-ary sets as well as M-ary FSK. We consider equienergy signals, i.e., $E_i = E_s\ \forall i$, and we choose a typical value $\gamma_s = 10$ dB, with $\gamma_s = E_s/N_0$ being the signal-to-noise ratio (SNR) per symbol. Also, we consider $M = 4$ and 8. Note that even for $M = 4$, searching for optimum correlation structures is a complicated and time consuming task.

1) Minimum SEP of M-ary FSK: One important property of $\underline{R}_s$ of FSK is that each element is given by $\rho_{i,j} = \mathrm{sinc}(2\Delta f_{i,j}\,T_s)$, with $\Delta f_{i,j}$ being the frequency spacing between the ith and jth tones [5, Section 3.1.6]. Although $\underline{R}_s$ involves $M\,(M-1)/2$ elements, we have a problem with only $M-1$ adjacent varying correlation coefficients, $\rho_{i,i+1}$, since the rest $(M-1)\,(M-2)/2$ can be easily obtained from $\rho_{i,i+1}$'s.

We start from the minimum value -0.21723 for all $\rho_{i,i+1}$'s and each one consecutively increases by a step of 0.0246 till a maximum value 0.25. We begin with $\rho_{1,2} = \rho_{2,3} = -0.21723$ and first vary $\rho_{3,4}$ from -0.21723 to 0.25. In each step, we mark the correlation matrix with an index value, e.g., the index 1 is set for $\rho_{1,2} = \rho_{2,3} = \rho_{3,4} = -0.21723$. Then, we increase $\rho_{2,3}$ according to the predefined step value and vary $\rho_{3,4}$ from -0.21723 to 0.25. After $\rho_{2,3}$ reaches 0.25, we increase $\rho_{1,2}$ and repeat the previous procedure for $\rho_{2,3}$ and $\rho_{3,4}$ till all three correlation coefficients take the maximum value 0.25. Each correlation coefficient gets 20 values. Every group of 20×20 values for P_{es} is shown in Fig. 1 as a half ellipsoid. As we can see, the smallest value for the SEP is at the eighth ellipsoid and specifically at the index 2989 of the correlation matrix

$$\underline{R}_s = \begin{bmatrix} 1 & -0.0451 & 0.0211 & -0.0208 \\ -0.0451 & 1 & 0.0041 & 0.0083 \\ 0.0211 & 0.0041 & 1 & -0.0205 \\ -0.0208 & 0.0083 & -0.0205 & 1 \end{bmatrix}$$

having the value $P_{es} = 2.17 \cdot 10^{-3}$. Note that the SEP for orthogonal 4FSK is $P_{es} = 2.24 \cdot 10^{-3}$.

A figure similar to Fig. 1 is Fig. 2 for equienergy 8-FSK, in which the correlation coefficients range from -0.144 to 0.0 with a predefined step value of 0.024. Each $\rho_{i,i+1}$ gets 6 values, and therefore, there are $6^7 = 279936$ points for the correlation matrix index. The lowest value for the SEP $P_{es} = 4.42 \cdot 10^{-3}$ can be found at the index 231940 for

$$\underline{R}_s = \begin{bmatrix} 1 & -0.0286 & 0.0145 & -0.0191 & 0.0144 & -0.0171 & 0.0188 & -0.0236 \\ -0.0286 & 1 & 0.0 & 0.0145 & -0.0097 & 0.0144 & -0.0171 & 0.0234 \\ 0.0145 & 0.0 & 1 & -0.0286 & 0.0145 & -0.0191 & 0.0284 & -0.0483 \\ -0.0191 & 0.0145 & -0.0286 & 1 & 0.0 & 0.0145 & -0.0191 & 0.0284 \\ 0.0144 & -0.0097 & 0.0145 & 0.0 & 1 & -0.0286 & 0.0284 & -0.0375 \\ -0.0171 & 0.0144 & -0.0191 & 0.0145 & -0.0286 & 1 & -0.0286 & 0.0427 \\ 0.0188 & -0.0171 & 0.0284 & -0.0191 & 0.0284 & -0.0286 & 1 & -0.0571 \\ -0.0236 & 0.0234 & -0.0483 & 0.0284 & -0.0375 & 0.0427 & -0.0571 & 1 \end{bmatrix}$$

whereas the SEP for orthogonal 8-FSK signaling is $P_{es} = 4.84 \cdot 10^{-3}$.

From both Figs. 1 and 2, we conclude that the SNR gain achieved using correlated FSK signaling offers a very slight advantage compared to orthogonal FSK signaling.

2) Minimum SEP of Arbitrary Correlated Signals: For arbitrarily correlated signaling, conclusions seem to be more obvious than for FSK, although $M\,(M-1)/2$ independently varying correlation coefficients are involved. From all simulation runs it became clear that the optimum set of $\rho_{i,j}$'s is

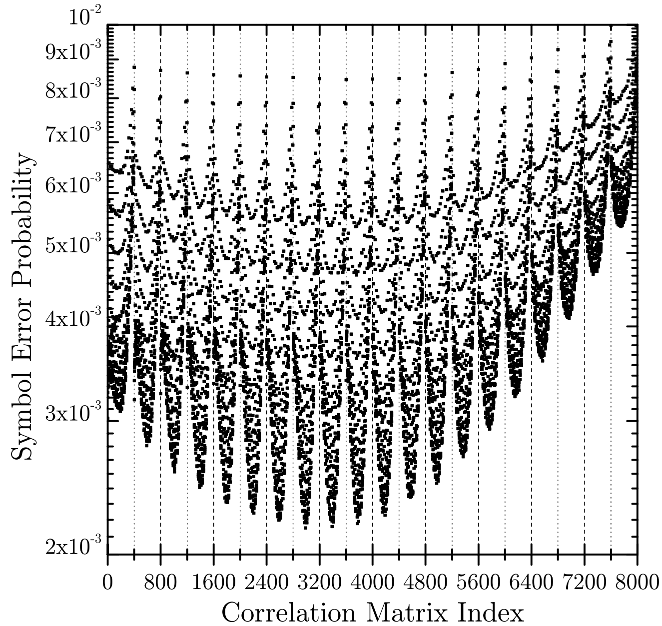

Fig. 1. SEP of equienergy 4-FSK as a function of correlation matrix index for $\gamma_s = 10$ dB and with correlation coefficients ranging from -0.2172 to 0.25.

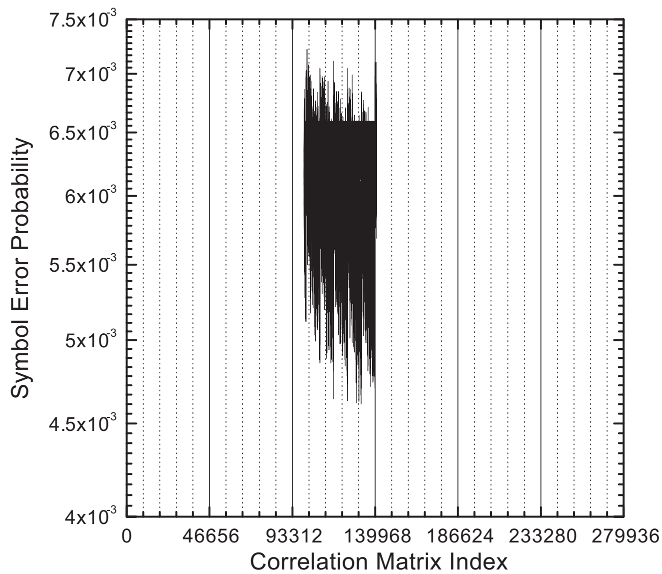

Fig. 2. SEP of equienergy 8-FSK as a function of correlation matrix index for $\gamma_s = 10$ dB and with correlation coefficients ranging from -0.144 to 0.0.

obtained for simplex signals, i.e., equicorrelated with $\rho_{i,j} = -1/(M-1)\ \forall i \neq j$.

III. Symbol Error Probability Analysis for Specific Correlation Structures

Starting with (4) and letting $x_j(i) \triangleq L_j(i) - L_i(i)$, we define the vector $\underline{x}(i)$ as $\underline{x}(i) \triangleq [x_1(i), \ldots, x_{i-1}(i), x_{i+1}(i), \ldots, x_M(i)]^T$ in order to study the joint statistics of its Gaussian elements. Using (3), the mean of $x_j(i)$ is $\mathbb{E}\langle x_j(i)\rangle = \rho_{i,j}\sqrt{E_i\,E_j} - (E_i + E_j)/2$, and denoting n_j as $n_j = \int_0^{T_s} n(t)\, s_j(t)\,\mathrm{d}t$, $L_j(i)$ can be rewritten as $L_j(i) = \rho_{i,j}\sqrt{E_i\,E_j} - E_j/2 + n_j$, for $i \neq j$, and $L_j(i) = E_i/2 + n_i$, for $i = j$. It can be easily shown that n_j is a zero mean Gaussian RV, with $\mathbb{E}\langle n_j\, n_k\rangle = \rho_{j,k}\sqrt{E_j\,E_k}\,N_0/2$, for $j \neq k$, and $\mathbb{E}\langle n_j\, n_k\rangle = E_j\,N_0/2$, for $j = k$. Also, for $j,k \neq i$, the covariance between $x_j(i)$ and $x_k(i)$ is $\mathrm{cov}[x_j(i), x_k(i)] = (\rho_{j,k}\sqrt{E_j\,E_k} - \rho_{k,i}\sqrt{E_k\,E_i} - \rho_{j,i}\sqrt{E_j\,E_i} + E_i)N_0/2$, for $j \neq k$, and $\mathrm{cov}[x_j(i), x_k(i)] = (E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i})N_0/2$, for $j = k$. The correlation coefficient $\epsilon_{j,k}(i)$ between $x_j(i)$ and $x_k(i)$ is therefore

$$\epsilon_{j,k}(i) = \frac{\rho_{j,k}\sqrt{E_j\,E_k} - \rho_{k,i}\sqrt{E_k\,E_i} - \rho_{j,i}\sqrt{E_j\,E_i} + E_i}{\sqrt{E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i}}\sqrt{E_k + E_i - 2\,\rho_{k,i}\sqrt{E_k\,E_i}}}. \quad (6)$$

We now consider the case where, $\forall\, j \neq k$, $\epsilon_{j,k}$ can be written as a product of two separate terms with respect to the indices j and k, e.g.,

$$\epsilon_{j,k}(i) = \nu_{j,i}\,\nu_{k,i}\,. \quad (7)$$

In order to follow a standard way of decomposing the Gaussian elements of $\underline{x}(i)$ having certain correlation properties into independent and identically distributed (i.i.d.) Gaussian RVs [13], we define $\Omega_j(i) \triangleq (E_j + E_i - 2\,\rho_{j,i}\sqrt{E_j\,E_i})\,N_0/2$ and $\mu_j(i) \triangleq -\rho_{j,i}\sqrt{E_j\,E_i} + (E_j + E_i)/2$. Now $x_j(i)$ can be expressed as

$$\begin{aligned} x_j(i) &= -\mu_j(i) + n_j - n_i \\ &= -\mu_j(i) + \sqrt{\Omega_j(i)}\left(\sqrt{1-\nu_{j,i}^2}\,\vartheta_j + \nu_{j,i}\,\vartheta_i\right), \end{aligned} \quad (8)$$

with $\vartheta_1, \vartheta_2, \ldots, \vartheta_M$ being i.i.d. Gaussian RVs, each having a $\mathcal{N}(0,1)$ distribution. By substituting (8) in (4), the probability of correct decision, under the hypothesis that the ith signal is transmitted, is

$$\begin{aligned} P_{c_i} &= \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{\vartheta_i^2}{2}\right)\left[\prod_{\substack{j=1\\ j\neq i}}^{M} \Pr\left[x_j(i) < 0 \,|\, \vartheta_i\right]\right]\mathrm{d}\vartheta_i \\ &= \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty}\left[\prod_{\substack{j=1\\ j\neq i}}^{M} Q\left(\frac{-\mu_j(i) + \nu_{j,i}\,\vartheta_i\sqrt{\Omega_j(i)}}{\sqrt{\Omega_j(i)}\sqrt{1-\nu_{j,i}^2}}\right)\right] \\ &\qquad \times \exp\left(-\frac{\vartheta_i^2}{2}\right)\mathrm{d}\vartheta_i\,, \end{aligned} \quad (9)$$

with $Q(x) = \mathrm{erfc}(x/\sqrt{2})/2$, $\mathrm{erfc}(\cdot)$ being the complementary error function. The SEP of coherently detected and correlated signals can be obtained by substituting (9) in (5). By comparing (9) with [2, eq. (4.80)], it is clear that our new result is less generic but much simpler, since it is in the form of a single integral. More importantly, the numerical evaluation of (9) is much less time consuming and complicated. Also, [2, eq. (4.80)] is limited to equienergy signals.

For binary signaling $(M = 2), \nu_{j,i}(i) = \nu_{k,i}(i) = 1$ $(i,j,k = 1$ and $2)$, and thus $x_j(i) = -\mu_j(i) + \sqrt{\Omega_j(i)}\,\vartheta_i$. By substituting $x_j(i)$ in P_{c_i} (first equality in (9)) yields

$P_{c_i} = Q(-\mu_j(i)/\sqrt{\Omega_j(i)})$. Therefore, the bit error probability $P_{eb} = 1 - (P_{c_1} + P_{c_2})/2$ is expressed as

$$P_{eb} = Q\left(\sqrt{\frac{E_1 + E_2 - 2\,\rho_{1,2}\sqrt{E_1\,E_2}}{2\,N_0}}\right), \quad (10)$$

which is in agreement with [4, eq. (B.33)]. The minimum value of P_{eb} is obtained for $\rho_{1,2} = \rho_{\min} = -1$. For $M \geq 2$ and in order for $\epsilon_{j,k}(i)$ to have the form presented in (7), a general solution exists when

$$\rho_{i,j} = \frac{\rho_i\,E_i + \rho_j\,E_j}{2\sqrt{E_i\,E_j}}, \quad (11)$$

where $|\rho_i| \leq 1$ is a function of the index i, e.g., $\rho_i = \rho^i\ \forall i$. For such a correlation coefficient, $\nu_{j,i} = 1/\sqrt{1+\zeta_{j,i}}$, where $\zeta_{j,i} = (E_j/E_i)\,(1-\rho_j)/(1-\rho_i)$. However, since $|\rho_{i,j}| \leq 1$, the following constraint on ρ_i and ρ_j exists: $|\rho_i\,E_i + \rho_j\,E_j| \leq 2\sqrt{E_i\,E_j}$. In addition, for $\rho_{i,j} = 0$, $\rho_i = 0\ \forall i$. Now based on (11), (9) can be expressed as

$$P_{c_i} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{\vartheta_i^2}{2}\right) \times\left[\prod_{\substack{j=1\\ j\neq i}}^{M} Q\left(-\sqrt{(1-\rho_i)\,\frac{E_i}{2\,N_0}\,\frac{1+\zeta_{j,i}}{\sqrt{\zeta_{j,i}}}} + \frac{\vartheta_i}{\sqrt{\zeta_{j,i}}}\right)\right]\mathrm{d}\vartheta_i\,. \quad (12)$$

Note that a simple form for ρ_i is $\rho_i = i\,\rho/M$ with $|\rho| \leq 1$.

A. *Orthogonal Signaling*

When all M signals are orthogonal to each other, $\rho_{i,j} = 0$ $\forall i \neq j$, implying that $\rho_i = \rho_j = 0$. When we further consider equienergy signals, i.e., $E_i = E_j = E_s\ \forall i,j$ ($P_{c_i} = P_c$), $P_{es} = 1 - P_c$ agrees with [5, eq. (8.41)].

B. *Equicorrelated Signaling*

For constant correlation signaling sets, $\rho_{j,i} = \rho\ \forall j \neq i$. Setting $\rho_i = \rho_j = \rho$ in (12), we get

$$P_{c_i} = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{\vartheta_i^2}{2}\right)\left[\prod_{\substack{j=1\\ j\neq i}}^{M} Q\left(-\sqrt{(1-\rho)\,\frac{E_i+E_j}{2\,N_0}} \times\sqrt{1+\frac{E_i}{E_j}} + \sqrt{\frac{E_i}{E_j}}\,\vartheta_i\right)\right]\mathrm{d}\vartheta_i\,. \quad (13)$$

In the Appendix it is proved that $\forall i\ P_{c_i}$ is maximized for

$$\rho_{\min} = -\frac{1}{M-1}, \quad (14)$$

and hence, this value of the correlation coefficient minimizes P_{es}. It is worth noticing that in case of equienergy and not necessarily equicorrelated signaling with correlation coefficients given as $\rho_{i,j} = (\rho_i + \rho_j)/2$, constant correlation signaling with $\rho_i = \rho_{\min}\ \forall i$ also minimizes P_{es}. This is the simplex signaling set with its SEP yielding from (5) and (13) as

$$P_{es} = (M-1)\,Q\left(\sqrt{\gamma_s\,(1-\rho_{\min})}\right) + \frac{1}{\sqrt{2\,\pi}}\sum_{k=2}^{M-1}(-1)^{k+1}\binom{M-1}{k}\int_{-\infty}^{\infty}\exp\left(-\frac{\vartheta^2}{2}\right) \times Q^k\left(\sqrt{2\,\gamma_s\,(1-\rho_{\min})} - \vartheta\right)\mathrm{d}\vartheta\,. \quad (15)$$

Note that for $M = 3$, (15) numerically agrees with the results in [14].

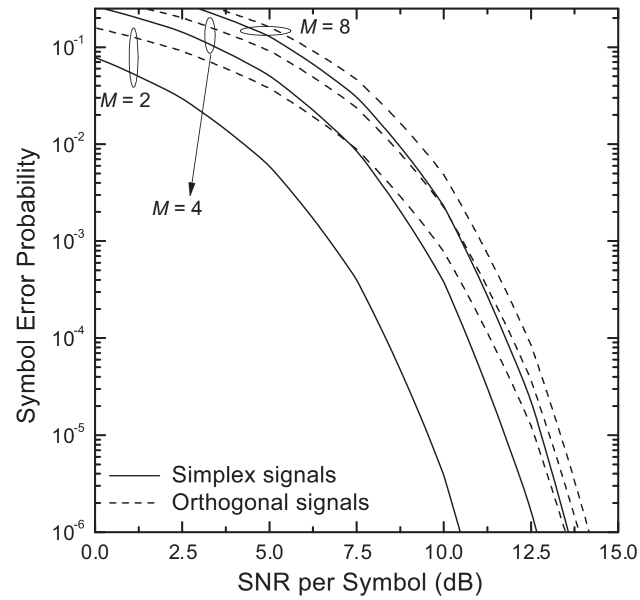

Fig. 3. SEP of simplex and orthogonal signaling sets as a function of received SNR per symbol.

Fig. 3 shows the SEP of simplex and orthogonal signals as a function of γ_s. As shown, simplex signaling indicates a better error performance than orthogonal signaling. Specifically, a constant SNR difference between simplex and orthogonal signaling that is $10\log_{10}[M/(M-1)]$ dB is observed [1, eq. (5-2-35)]. However, as M increases, this difference tends to vanish.

IV. Generation of Multilevel Signaling Sets

Given a set of M complex orthonormal waveforms $\phi_1(t)$, $\phi_2(t), \ldots, \phi_M(t)$ over $[0, T_s)$, we can easily generate a signaling set $s_1(t), s_2(t), \ldots, s_M(t)$ with any correlation structure $\int_0^{T_s} s_i(t)\,s_j^*(t)\,\mathrm{d}t = 2\sqrt{E_i\,E_j}\,\rho_{i,j}$, where $(\cdot)^*$ denotes the complex conjugate. The approach is as follows:

- Let $\underline{\phi}(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_M(t)]^T$, with $\int_0^{T_s}\underline{\phi}(t) \times\underline{\phi}^H(t)\,\mathrm{d}t = \underline{I}_M$ owing to orthonormality ($(\cdot)^H$ denotes the Hermitian operator).
- Let $\underline{s}(t) = [s_1(t), s_2(t), \ldots, s_M(t)]^T$ and let $\underline{\Sigma}_s = \int_0^{T_s}\underline{s}(t)\,\underline{s}^H(t)\,\mathrm{d}t$ be the correlation matrix of the signaling set. The element in the ith row and the jth column of $\underline{\Sigma}_s$ is $[\underline{\Sigma}_s]_{i,j} = 2\sqrt{E_i\,E_j}\,\rho_{i,j}$. Note that $\underline{\Sigma}_s$ is a Hermitian matrix.

If we express

$$\underline{s}(t) = \underline{A}\,\underline{\phi}(t)\,, \tag{16}$$

with $\underline{A}\,\underline{A}^H = \underline{\Sigma}_s$, then $\underline{s}(t)$ has the desired correlation matrix $\underline{\Sigma}_s$.

A. Transorthogonal Signals

In the case of transorthogonal signals, $\underline{\Sigma}_s$ is called the Mth-order intraclass correlation matrix [5, Section 9.7.4.2]. The eigenvalues of $\underline{\Sigma}_s$ are $1-\rho$ and $1+(M-1)\,\rho$, with multiplicities $M-1$ and 1, respectively. Since $\underline{\Sigma}_s$ is positive semi-definite (non-negative eigenvalues), $\rho \geq \rho_{\min} = -1/(M-1)$. Using the well known eigendecomposition method, $\underline{\Sigma}_s$ can be decomposed as $\underline{\Sigma}_s = \underline{U}\,\underline{D}\,\underline{U}^H$, where $\underline{U}$ is an $M \times M$ unitary matrix containing the orthonormal eigenvectors of $\underline{\Sigma}_s$ in its columns and $\underline{D}$ the corresponding diagonal matrix of eigenvalues, given by $\underline{D} = \mathrm{diag}(1-\rho, 1-\rho, \ldots, 1-\rho, 1+(M-1)\,\rho)$. The matrix $\underline{A}$ can be obtained as $\underline{A} = \underline{U}\sqrt{\underline{D}}$. We must also mention that instead of using the eigendecomposition method, simplex signals can be alternatively constructed as described in [1, eq. (4-3-35)].

B. Signals with Arbitrary Correlation

In general, and except for the case of the transorthogonal signaling set, it is very difficult to derive the eigenvalues and the eigenvectors of $\underline{\Sigma}_s$ in closed form using the eigendecomposition method. Thus, as a general method, LU decomposition [15] can be followed. In this method, we can split $\underline{\Sigma}_s$ as $\underline{\Sigma}_s = \underline{L}\,\underline{D}'\,\underline{L}^H$, where all the main diagonal entries of the lower triangular matrix $\underline{L}$ are equal to one ($\underline{L}^H$ is upper triangular) and each of the main diagonal entries of the diagonal matrix $\underline{D}'$ is equal to the corresponding leading principal minor of $\underline{\Sigma}_s$. The matrix $\underline{A}$ can be obtained as $\underline{A} = \underline{L}\sqrt{\underline{D}'}$.

V. Conclusions

We have studied coherent detection for multilevel correlated signaling sets over AWGN. Correlation structures that minimize the SEP of multilevel FSK and arbitrarily correlated signaling sets have been investigated via computer simulations. It is found that the SNR gain achieved by correlated FSK signaling offers a very slight advantage. Also, it has become clear that the correlation matrix that minimizes the SEP in case of arbitrarily correlated signaling sets is the equicorrelated one. Moreover, a general and analytical expression for the SEP has been derived for specific correlation structures. Based on eigendecomposition or LU decomposition, generic methods for constructing a correlated signaling set for any correlation matrix under consideration is also provided.

Appendix
Proof that $\rho_{\min} = -1/(M-1)$ Minimizes SEP

Taking the first derivative of P_{c_i} in (13) with respect to ρ yields

$$\frac{\partial P_{c_i}}{\partial \rho} = \frac{-1}{4\,\pi} \sum_{\substack{k=1\\k\neq i}}^{M} \sqrt{\xi_{i,k}\,\frac{1+\lambda_{i,k}}{1-\rho}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\right.$$
$$\times \left.\left[-\sqrt{(1-\rho)\,\xi_{i,k}}\,\sqrt{1+\lambda_{i,k}} + \sqrt{\lambda_{i,k}}\,\vartheta_i\right]^2 - \frac{\vartheta_i^2}{2}\right\}$$
$$\times \left[\prod_{\substack{j=1\\j\neq i,k}}^{M} Q\left(-\sqrt{(1-\rho)\,\xi_{i,j}}\,\sqrt{1+\lambda_{i,j}} + \sqrt{\lambda_{i,j}}\,\vartheta_i\right)\right] \mathrm{d}\vartheta_i\,, \tag{A-1}$$

with $\lambda_{i,j} = E_i/E_j$ and $\xi_{i,j} = (E_i + E_j)/(2\,N_0)$. Since the argument of the above integral is always positive for $|\rho| < 1$, it becomes obvious that $\partial P_{c_i}/\partial\rho < 0$, i.e., P_{c_i} is a strictly decreasing function of ρ. Hence, based on (5), the minimum P_{es} is obtained for minimum ρ, as explained in Section IV-A, and is given by $\rho_{\min} = -1/(M-1)$.

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