# A Trivariate Nakagami-$m$ Distribution with Arbitrary Covariance Matrix and Applications to Generalized-Selection Diversity Receivers

Kostas Peppas, *Member, IEEE*, and Nikos C. Sagias, *Member, IEEE*

***Abstract*—This paper deals with a trivariate Nakagami-$m$ distribution derived from the diagonal elements of a Wishart matrix. For this distribution, infinite series representations for its probability density and cumulative distribution functions are derived having an arbitrary covariance matrix and integer-order fading parameters. Moreover, upper bounds on the error resulting from truncating the infinite series are obtained. Based on the derived formulas, the performance of triple-branch generalized-selection combining (GSC) receivers is analyzed. For this type of receivers, the outage and the average bit error probability for a variety of modulation schemes are analytically obtained. The performance of GSC receivers is compared to that of conventional selection and maximal-ratio diversity schemes. In order to check the accuracy and convergence of the derived formulas, various performance evaluation results are presented and compared to equivalent simulation ones.**

***Index Terms*—Average bit error probability (ABEP), correlated statistics, diversity, generalized-selection diversity, maximal-ratio combining (MRC), multichannel receivers, Nakagami-$m$ fading, order statistics, outage probability, selection diversity, stochastic models, Wishart matrix.**

## I. INTRODUCTION

THE Nakagami-$m$ distribution is used in modeling various propagation channels which are characterized by multipath scattering with relatively large delay-time spreads and different clusters of reflected waves [1]–[3]. Of particular interest is the multivariate Nakagami-$m$ distribution, which is able to further modeling correlation among wireless channels [1], [3]–[7]. Multichannel receivers operating in such channels improve wireless link performance using efficient diversity schemes [8]–[16]. Among the most known are the maximal-ratio combining (MRC) and the selection combining (SC) schemes. Also, a combination of MRC and SC, identified as generalized-selection combining (GSC), has been proposed to bridge the performance and complexity gaps between MRC and SC.

Various works study the performance of GSC receivers over correlated Nakagami-$m$ fading channels [16]–[22]. For example in [16], an analysis of GSC receivers over equicorrelated slow frequency-nonselective Nakagami-$m$ fading channel has been presented. Moreover in [19], [20], by assuming correlated Nakagami-$m$ fading with positive integer-order values for the fading parameters, the performance of GSC and threshold-based hybrid SC/MRC receivers has been analyzed. More specifically in [19], a more general model than the equicorrelated has been considered, while in [20], Green's matrix approximations have been used for studying arbitrary correlation structures.

Besides, various research findings have been reported in the statistics research literature, but have not been considered by researchers working on wireless communications theory yet. Some of them concern joint chi-square distributions derived from the diagonal elements of the Wishart matrix [23]–[28]. Although the derivation of the joint moment-generating function (MGF) is straightforward, the extraction of the associated probability density function (PDF) or cumulative distribution function (CDF) is quite complicated. In [23], a closed-form expression for the distribution of the diagonal elements of the Wishart matrix has been obtained, but that result seems to be difficult to be numerically evaluated. In [28], [29], expansions for the PDF of a trivariate chi-square distribution have been presented in terms of rapidly convergent infinite sums which are simple for numerical evaluation.

Based on [29], a trivariate Nakagami-$m$ distribution is introduced in this paper. It is derived from the diagonal elements of the complex Wishart matrix, having integer-order fading parameters and an arbitrary covariance matrix. For this distribution, infinite series representations for its PDF and CDF as well as upper bounds on the error resulting from truncating the infinite series are extracted. The derived expressions are used to assess the performance of triple-branch GSC receivers in terms of the outage and the average bit error probability (ABEP). Performance comparisons with conventional triple-branch MRC and SC receivers are also presented.

The paper is organized as follows: In Section II, the trivariate Nakagami-$m$ distribution is introduced. In Section III, the theoretical results presented in Section II are applied to analyze the performance of triple-branch GSC receivers. In Section IV, numerical and computer simulation results are illustrated and compared, while the paper concludes with a summary given in Section V.

Next, the following notations are used: $(\cdot)^T$ for the transpose, $(\cdot)^{-1}$ for the inverse, $(\cdot)^H$ for the Hermitian transpose, $\det(\cdot)$ for the determinant, $(\cdot)^*$ for the complex conjugate, $\mathrm{diag}(\cdot)$ for the diagonal elements, $\mathbb{E}\langle\cdot\rangle$ for the expectation operator, $\mathbb{L}^{-1}[\cdot;\cdot]$ for the inverse Laplace transform, and $\Re(\cdot)$ and $\Im(\cdot)$ the real and imaginary parts operators, respectively.

Paper approved by M.-S. Alouini, the Editor for Modulation and Diversity Systems of the IEEE Communications Society. Manuscript received June 4, 2007; revised December 6, 2007 and March 9, 2008.

This paper was presented in part at the International Symposium on Wireless Pervasive Computing (ISWPC'08), Santorini, Greece, May 2008.

K. Peppas is with the Laboratory of Mobile Communications, Institute of Informatics and Telecommunications, National Centre for Scientific Research–"Demokritos," Patriarhou Grigoriou and Neapoleos, Agia Paraskevi, 15310, Athens, Greece (e-mail: peppas@telecom.ntua.gr).

N. C. Sagias is with the Department of Telecommunications Science & Technology, School of Applied Sciences and Technology, University of Peloponnese, Karaiskaki Str., 22100, Tripolis, Greece (e-mail: nsagias@ieee.org).

Digital Object Identifier 10.1109/TCOMM.2009.07.070071

## II. The Trivariate Nakagami-$m$ Distribution

After first recalling same basic issues related to the Wishart matrix, infinite series representations of the joint PDF and CDF of the trivariate Nakagami-$m$ distribution are provided, which are the key results of this section. Additionally, upper bounds on the error resulting from truncating the infinite series are derived.

### A. Preliminaries

Let $\mathbf{Q}_p = [X_{1,p}, X_{2,p}, X_{3,p}]^T$ be the $p$th sample of a three-dimensional zero-mean complex Gaussian random process ($p = 1, 2, \ldots, m$). These processes are considered to be mutually independent and identically distributed (id) having a covariance matrix

$$\mathbf{\Sigma} = 2 \begin{bmatrix} \sigma_1^2 & c_{12} & c_{13} \\ c_{12} & \sigma_2^2 & c_{23} \\ c_{13} & c_{23} & \sigma_3^2 \end{bmatrix} \tag{1}$$

with $\sigma_\ell^2 = \mathbb{E}\langle |X_{\ell,p}|^2 \rangle /2$ and $c_{\ell,\ell'} = \mathbb{E}\langle X_{\ell,p} X_{\ell',p}^* \rangle /2, \forall \ell \neq \ell'$ ($\ell, \ell' = 1, 2$, and 3). By defining a six-dimensional sample vector as $\mathbf{W}_p = [\Re(X_{1,p}), \Im(X_{1,p}), \Re(X_{2,p}), \Im(X_{2,p}), \Re(X_{3,p}), \Im(X_{3,p})]^T$ and setting the crosscorrelation terms between real and imaginary parts equal to zero, $\mathbb{E}\langle \Re\{X_{\ell,p}\} \Im\{X_{\ell',p}\} \rangle = 0$ $\forall \ell, \ell' = 1, 2, 3$, the joint PDF of $\mathbf{W}_p$ is

$$f_{\mathbf{W}_p}(\mathbf{W}_p) = \frac{1}{\sqrt{(2\pi)^6 \det(\mathbf{C})}} \exp\left(-\frac{1}{2} \mathbf{W}_p^T \mathbf{C}^{-1} \mathbf{W}_p\right) \tag{2}$$

where $\mathbf{C} = \mathbb{E}\langle \mathbf{W}_p \mathbf{W}_p^T \rangle$ having the following structure

$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & 0 & c_{12} & 0 & c_{13} & 0 \\ 0 & \sigma_1^2 & 0 & c_{12} & 0 & c_{13} \\ c_{12} & 0 & \sigma_2^2 & 0 & c_{23} & 0 \\ 0 & c_{12} & 0 & \sigma_2^2 & 0 & c_{23} \\ c_{13} & 0 & c_{23} & 0 & \sigma_3^2 & 0 \\ 0 & c_{13} & 0 & c_{23} & 0 & \sigma_3^2 \end{bmatrix}. \tag{3}$$

The inverse of $\mathbf{C}$ is

$$\mathbf{C}^{-1} = \begin{bmatrix} a_1 & 0 & b_1 & 0 & b_3 & 0 \\ 0 & a_1 & 0 & b_1 & 0 & b_3 \\ b_1 & 0 & a_2 & 0 & b_2 & 0 \\ 0 & b_1 & 0 & a_2 & 0 & b_2 \\ b_3 & 0 & b_2 & 0 & a_3 & 0 \\ 0 & b_3 & 0 & b_2 & 0 & a_3 \end{bmatrix} \tag{4}$$

where

$$a_1 = \frac{\sigma_2^2 \sigma_3^2 - c_{23}^2}{\sqrt{\det(\mathbf{C})}}, \qquad b_1 = \frac{\sigma_3^2 c_{12} - c_{13} c_{23}}{\sqrt{\det(\mathbf{C})}} \tag{5a}$$

$$a_2 = \frac{\sigma_1^2 \sigma_3^2 - c_{13}^2}{\sqrt{\det(\mathbf{C})}}, \qquad b_2 = \frac{\sigma_1^2 c_{23} - c_{12} c_{13}}{\sqrt{\det(\mathbf{C})}} \tag{5b}$$

$$a_3 = \frac{\sigma_1^2 \sigma_2^2 - c_{12}^2}{\sqrt{\det(\mathbf{C})}}, \qquad b_3 = \frac{\sigma_2^2 c_{13} - c_{12} c_{23}}{\sqrt{\det(\mathbf{C})}} \tag{5c}$$

while the determinant of $\mathbf{C}$ is $\det(\mathbf{C}) = 2^{-6} [\det(\mathbf{\Sigma})]^2$.

If we define vector $\mathbf{X}_\ell = [\Re(X_{\ell,1}), \Im(X_{\ell,1}), \Re(X_{\ell,2}), \Im(X_{\ell,2}), \ldots, \Re(X_{\ell,m}), \Im(X_{\ell,m})]$, then its norm is given by $R_\ell^2 = ||\mathbf{X}_\ell||^2 = \sum_{p=1}^m [\Re^2(X_{\ell,p}) + \Im^2(X_{\ell,p})]$, which essentially denotes the diagonal elements of the complex Wishart matrix, $\mathbf{S} = \sum_{p=1}^m \mathbf{Q}_p \mathbf{Q}_p^H$. It is obvious that $R_\ell$ is a Nakagami-$m$ random variable (RV), with an integer-order fading parameter $m$ and average power $\Omega_\ell = \mathbb{E}\langle R_\ell^2 \rangle = 2\, m\, \sigma_\ell^2$, having PDF

$$f_{R_\ell}(r) = \frac{2\, m^m}{\Omega_\ell^m\, (m-1)!}\, r^{2m-1} \exp\left(-\frac{m}{\Omega_\ell} r^2\right). \tag{6}$$

Also, the $n$th-order moment of $R_\ell$ [1, eq. (17)] is given by $\mathbb{E}\langle R_\ell^n \rangle = (\Omega_\ell/m)^{n/2}\, \Gamma(m + n/2)/(m-1)!$, with $\Gamma(\cdot)$ being the gamma function [30, eq. (8.310/1)].

### B. Joint PDF

According to [29], there are two cases which should be taken into consideration for the trivariate Nakagami-$m$ PDF. The first one is for $m = 1$ and the second one for $m = 2, 3, \ldots$ Since the first case (Rayleigh fading) has been already studied in [31], next we are exclusively interested in the second one. Using (2), the joint PDF of $R_\ell$ can be extracted as [29, eq. (11)]

$$f_{R_1,R_2,R_3}(r_1, r_2, r_3) = \int\limits_{|\mathbf{X}_1|=r_1} \int\limits_{|\mathbf{X}_2|=r_2} \int\limits_{|\mathbf{X}_3|=r_3} f_{\mathbf{W}_p}(\mathbf{W}_p)\, \mathrm{d}\mathbf{X}_1\, \mathrm{d}\mathbf{X}_2\, \mathrm{d}\mathbf{X}_3 \tag{7}$$

where $\int_{|\mathbf{X}_\ell|=r_\ell}$ denotes integration over the surface of a $2m$-dimensional sphere of radius $r_\ell$. The triple integral in (7) has been solved in [29], and thus, the trivariate Nakagami-$m$ PDF with integer-order fading parameters and an arbitrary covariance matrix (with entries $\mathbf{R}_{\ell,\ell'} = m\, \mathbf{\Sigma}_{\ell,\ell'}^2$) can be expressed as[1]

$$f_{R_1,R_2,R_3}(r_1, r_2, r_3) = \frac{\exp\left[-\left(a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2\right)/2\right]}{(m-1)\, \left[\det(\mathbf{\Sigma}/2)\right]^m\, (b_1 b_2 b_3)^{m-1}} \times r_1 r_2 r_3 \sum_{k=m-1}^{\infty} k\, (-1)^{k-m+1} \binom{m+k-2}{2m-3} \times I_k(b_1 r_1 r_2)\, I_k(b_2 r_2 r_3)\, I_k(b_3 r_1 r_3) \tag{8}$$

where $I_k(\cdot)$ denotes the $k$th-order modified Bessel function of the first kind [30, Section 8.406]. The power correlation coefficient between $\ell$th and $\ell'$th channels, defined as[2] $\rho_{\ell\ell'} = \mathrm{cov}(R_\ell^2, R_{\ell'}^2)/\left[\sqrt{\mathrm{var}(R_\ell^2)}\sqrt{\mathrm{var}(R_{\ell'}^2)}\right]$, can be easily expressed as a function of $c_{\ell\ell'}$ as $c_{\ell\ell'} = \sqrt{\rho_{\ell\ell'}\, \Omega_\ell\, \Omega_{\ell'}/(4\, m^2)}$. Hence, all parameters given by (5) can be reexpressed in terms of parameters of interest in wireless communications, such as $\rho_{\ell\ell'}$'s, $m$, and $\Omega_\ell$'s, as follows

$$a_1 = \frac{2m}{T\, \Omega_1}\, (1 - \rho_{23}), \qquad b_1 = \frac{-2\, m/T}{\sqrt{\Omega_1\, \Omega_2}} \left(\sqrt{\rho_{12}} - \sqrt{\rho_{23}\, \rho_{13}}\right) \tag{9a}$$

$$a_2 = \frac{2m}{T\, \Omega_2}\, (1 - \rho_{13}), \qquad b_2 = \frac{-2\, m/T}{\sqrt{\Omega_2\, \Omega_3}} \left(\sqrt{\rho_{23}} - \sqrt{\rho_{12}\, \rho_{13}}\right) \tag{9b}$$

$$a_3 = \frac{2m}{T\, \Omega_3}\, (1 - \rho_{12}), \qquad b_3 = \frac{-2\, m/T}{\sqrt{\Omega_1\, \Omega_3}} \left(\sqrt{\rho_{13}} - \sqrt{\rho_{12}\, \rho_{23}}\right) \tag{9c}$$

and $\det(\mathbf{\Sigma}) = T \prod_{\ell=1}^3 \Omega_\ell/m$ with $T = 1 - (\rho_{12} + \rho_{23} + \rho_{13}) + 2\sqrt{\rho_{12}\, \rho_{23}\, \rho_{13}}$.

[1]Note that (8) agrees with a parallel and independent result [32, eq. (19)].

[2]As it is well known, $\rho_{\ell\ell'}$ is related to the correlation coefficient of the underlying real Gaussian processes, $\varrho_{\ell\ell'}$, as $\rho_{\ell\ell'} = \varrho_{\ell\ell'}^2$.

*1) Uncorrelated:* In case where the three channels are uncorrelated, i.e., $\rho_{\ell,\ell'} = 0\ \forall \ell \neq \ell'$ ($T = 1$), $a_\ell = 2\,m/\Omega_\ell$ and $b_\ell = 0\ \forall \ell$. Based on the following power series expansion for $I_k(\cdot)$'s

$$\lim_{b_\ell \to 0} \frac{I_k\left(b_\ell\, r_\ell\, r_{\ell'}\right)}{b_\ell^{m-1}} = \begin{cases} \frac{1}{(m-1)!}\left(\frac{r_\ell\, r_{\ell'}}{2}\right)^{m-1}, & k = m-1; \\ 0, & k > m-1, \end{cases} \tag{10}$$

all terms except for $k = m-1$ vanish in the sum in (8), reducing $f_{R_1,R_2,R_3}(r_1, r_2, r_3)$ to a product of three independent marginal PDFs, i.e., $f_{R_1,R_2,R_3}(r_1, r_2, r_3) = \prod_{\ell=1}^{3} f_{R_\ell}(r_\ell)$.

*2) Constant correlation:* In case of constant correlation among the three channels, i.e., $\rho_{\ell,\ell'} = \rho\ \forall \ell \neq \ell'$ ($T = 1 - 3\,\rho + 2\,\rho^{3/2}$), $a_\ell = 2\,m\,(1-\rho)/\Omega_\ell$ and

$$b_1 = \frac{2\,m\,\tau}{\sqrt{\Omega_1\,\Omega_2}}, \quad b_2 = \frac{2\,m\,\tau}{\sqrt{\Omega_2\,\Omega_3}}, \text{ and } b_3 = \frac{2\,m\,\tau}{\sqrt{\Omega_1\,\Omega_3}}$$

with $\tau = \sqrt{\rho}/(2\,\rho - \sqrt{\rho} - 1)$.

*3) Exponential correlation:* In case of exponential correlation among the three channels, i.e., $\rho_{\ell,\ell'} = \rho^{|\ell-\ell'|}$ $\forall \ell, \ell'$ ($T = (1-\rho)^2$), $a_i = 2\,m/[\Omega_i\,(1-\rho)]$ ($i = 1$ and 3), $a_2 = (2\,m/\Omega_2)\,(1+\rho)/(1-\rho)$ and $b_j = -(2\,m/\sqrt{\Omega_j\,\Omega_{j+1}})\,\sqrt{\rho}/(1-\rho)$ ($j = 1$ and 2), $b_3 = 0$. Note that using (10) and setting the above specific form of parameters to (8), $f_{R_1,R_2,R_3}(r_1, r_2, r_3)$ reduces to [6, eq. (3)].

### C. Joint CDF

Using the PDF expression given by (8), the corresponding joint CDF of $R_\ell$ can be calculated as $F_{R_1,R_2,R_3}(r_1, r_2, r_3) = \int_0^{r_1}\int_0^{r_2}\int_0^{r_3} f_{R_1,R_2,R_3}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Using an infinite series representation for Bessel functions [30, eq. (8.445)] in (8) and after performing some straightforward algebraic manipulations, yields

$$\begin{aligned} F_{R_1,R_2,R_3}(r_1, r_2, r_3) = &\frac{T^{2\,m}}{m-1} \\ &\times \sum_{k=m-1}^{\infty} \sum_{l_1,l_2,l_3=0}^{\infty} k\,(-1)^{k-m+1} \binom{m+k-2}{2m-3} \\ &\times \prod_{\ell=1}^{3} \frac{\omega_\ell^{2\,l_\ell+k+1-m}\, n_\ell!}{\psi_\ell^{n_\ell+1}\, l_\ell!\,(l_\ell+k)!} \\ &\times \left[1 - \exp\left(-\frac{m\,\psi_\ell}{T\,\Omega_\ell}\, r_\ell^2\right) \sum_{p=0}^{n_\ell} \frac{1}{p!}\left(\frac{m\,\psi_\ell}{T\,\Omega_\ell}\, r_\ell^2\right)^p\right] \end{aligned} \tag{11}$$

with $n_1 = l_1 + l_3 + k$, $n_2 = l_1 + l_2 + k$, and $n_3 = l_2 + l_3 + k$. Also, $\psi_\ell$'s and $\omega_\ell$'s are defined as $\psi_1 = 1 - \rho_{23}$, $\psi_2 = 1 - \rho_{13}$, $\psi_3 = 1 - \rho_{12}$ and $\omega_1 = -\sqrt{\rho_{12}} + \sqrt{\rho_{23}\,\rho_{13}}$, $\omega_2 = -\sqrt{\rho_{23}} + \sqrt{\rho_{12}\,\rho_{13}}$, $\omega_3 = -\sqrt{\rho_{13}} + \sqrt{\rho_{12}\,\rho_{23}}$. Note that (11) consists only of elementary functions.

### D. Truncation Error

In order to find a simple bound for the truncation error of the CDF series in (11), we follow the method presented in [31, Section II.B]. Assume that the series in (11) are limited to $L_0$, $L_1$, $L_2$, and $L_3$ terms in indexes $k$, $l_1$, $l_2$, and $l_3$, respectively. The remaining terms constitute the truncation error. Based on the fact that $1 - \exp(x)\sum_{p=0}^{n_\ell} x^p/p! \leq 1$, the truncation error of (11) can therefore be upper bounded by

$$\begin{aligned} |E_T(L_0, L_1, L_2, L_3)| \leq &\sum_{k=L_0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_3=0}^{\infty} H(k, l_1, l_2, l_3) \\ &+ \sum_{k=m-1}^{L_0-1} \sum_{l_1=L_1}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_3=0}^{\infty} H(k, l_1, l_2, l_3) \\ &+ \sum_{k=m-1}^{L_0-1} \sum_{l_1=0}^{L_1-1} \sum_{l_2=L_2}^{\infty} \sum_{l_3=0}^{\infty} H(k, l_1, l_2, l_3) \\ &+ \sum_{k=m-1}^{L_0-1} \sum_{l_1=0}^{L_1-1} \sum_{l_2=0}^{L_2-1} \sum_{l_3=L_3}^{\infty} H(k, l_1, l_2, l_3) \end{aligned} \tag{12}$$

with

$$\begin{aligned} H(k, l_1, l_2, l_3) = &\frac{T^{2\,m}}{m-1} \binom{m+k-2}{2m-3} \binom{n_1}{l_3} \binom{n_2}{l_1} \\ &\times \binom{n_3}{l_2} \prod_{\ell=1}^{3} \frac{|\omega_\ell|^{2\,l_\ell+k+1-m}}{\psi_\ell^{n_\ell+1}}. \end{aligned} \tag{13}$$

As an example, we consider the constant correlation model, i.e., $\rho_{\ell\ell'} = \rho\ \forall \ell \neq \ell'$ ($\psi_\ell = 1 - \rho$ and $\omega_\ell = -\sqrt{\rho} + \rho$), with $m = 2$, 4 and $\rho = 0.1$, 0.5, 0.7. Setting $r_\ell^2/\Omega_\ell = r^2/\Omega\ \forall \ell$ and assuming $L_0 = L_1 = L_2 = L_3$, Table I summarizes the number of the terms required in (11) to achieve a ratio $|E_T|/F_{R_1,R_2,R_3}(r, r, r) < 10^{-3}$. As shown, the convergence rate depends strongly on $m$ and $\rho$. Specifically, the higher the $m$ and/or $\rho$ are, the more terms are needed. However, as $\rho$ approaches to unity, the bound in (12) becomes loose similarly to [4], [6]. Moreover, for fixed $m$ and $\rho$, as $r^2/\Omega$ increases, less terms are needed in the CDF series to achieve the target ratio.

TABLE I
NUMBER OF REQUIRED TERMS FOR CONVERGENCE OF (11) FOR THE CONSTANT CORRELATION MODEL TO ACHIEVE A TARGET RATIO $|E_T|/F_{R_1,R_2,R_3}(r, r, r) < 10^{-3}$

| | $r^2/\Omega = 0.3$ | | $r^2/\Omega = 1$ | | $r^2/\Omega = 3$ | |
|---|---|---|---|---|---|---|
| | $m = 2$ | $m = 4$ | $m = 2$ | $m = 4$ | $m = 2$ | $m = 4$ |
| $\rho = 0.1$ | 6 | 9 | 5 | 7 | 4 | 6 |
| $\rho = 0.5$ | 15 | 21 | 12 | 15 | 10 | 13 |
| $\rho = 0.7$ | 23 | 38 | 20 | 29 | 10 | 13 |

## III. TRIPLE-BRANCH GSC RECEIVERS

In this section, the performance of triple-branch GSC receivers operating over arbitrarily correlated Nakagami-$m$ fading channels is analyzed, utilizing known issues from the order statistics theory. The MGF of the GSC output signal-to-noise ratio (SNR) per symbol is analytically obtained and used to evaluate both the ABEP for various signalling constellations as well as the outage probability following the well-known MGF-based approach [3].

### A. System Model

We consider a GSC($K$,3) ($K = 1$, 2, 3) receiver operating over an arbitrary correlated Nakagami-$m$ multipath fading environment with not necessarily id channel statistics. According to GSC($K$,3) scheme, the $K$ strongest branches having the highest instantaneous SNRs are selected among the three available and appropriately combined. The GSC($K$,3) reception is equivalent to MRC reception if all three branches are combined ($K = 3$), while it is equivalent to SC reception if only one out of the three branches is selected ($K = 1$).

The baseband received signal at the $\ell$th diversity branch is $\zeta_\ell = z\,R_\ell + w_\ell$ where $z$ is the transmitted symbol with energy $E_s = \mathbb{E}\langle|z|^2\rangle$, $R_\ell$ is the Nakagami-$m$ distributed fading envelope, and $w_\ell$ is the additive white Gaussian noise with a single-sided power spectral density $N_0$. The noise components are assumed to be statistically independent of the signal and uncorrelated to each other. Moreover, all three channels are considered as slowly time varying, and thus, their characteristics are perfectly known to the receiver.

### B. SNR Joint Statistics

The instantaneous SNR per symbol $\gamma_\ell = R_\ell^2\,E_s/N_0$ in the $\ell$th input branch is an Erlang RV with $\overline{\gamma}_\ell = \Omega_\ell\,E_s/N_0$ being the corresponding average input SNR per symbol. By applying the following RVs transformation $R_\ell = \sqrt{\Omega_\ell\,\gamma_\ell/\overline{\gamma}_\ell}$ in (8), the joint PDF of $\gamma_1, \gamma_2, \gamma_3$ becomes

$$f_{\gamma_1,\gamma_2,\gamma_3}(\gamma_1,\gamma_2,\gamma_3) = \frac{2^{3\,(m-1)}\,\exp\left(-\frac{1}{2}\sum_{\ell=1}^{3}\widetilde{a}_\ell\,\gamma_\ell\right)}{(m-1)\,\left[\det\left(\widetilde{\mathbf{\Sigma}}\right)\right]^m\left(\widetilde{b}_1\,\widetilde{b}_2\,\widetilde{b}_3\right)^{m-1}} \times \sum_{k=m-1}^{\infty} k\,(-1)^{k-m+1}\binom{m+k-2}{2m-3} I_k\left(\widetilde{b}_1\sqrt{\gamma_1\,\gamma_2}\right) \times I_k\left(\widetilde{b}_2\sqrt{\gamma_2\,\gamma_3}\right)\,I_k\left(\widetilde{b}_3\sqrt{\gamma_1\,\gamma_3}\right). \tag{14}$$

The parameters $\widetilde{a}_\ell$ and $\widetilde{b}_\ell$ as well as the determinant $\det(\widetilde{\mathbf{\Sigma}})$ can be easily derived from corresponding $a_\ell$ and $b_\ell$, and $\det(\mathbf{\Sigma})$, just replacing $\Omega_\ell$ with $\overline{\gamma}_\ell$ $\forall\ell$ in (9). Similarly to Section II-C, the corresponding CDF to $f_{\gamma_1,\gamma_2,\gamma_3}(\gamma_1,\gamma_2,\gamma_3)$ can be easily found to be

$$F_{\gamma_1,\gamma_2,\gamma_3}(\gamma_1,\gamma_2,\gamma_3) = \frac{T^{2\,m}}{m-1}\sum_{k=m-1}^{\infty}\sum_{l_1,l_2,l_3=0}^{\infty} k\,(-1)^{k-m+1} \times \binom{m+k-2}{2m-3}\prod_{\ell=1}^{3}\frac{\omega_\ell^{2l_\ell+k+1-m}\,n_\ell!}{\psi_\ell^{1+n_\ell}\,l_\ell!\,(l_\ell+k)!} \times \left[1-\exp\left(-\frac{m\,\psi_\ell}{T\,\overline{\gamma}_\ell}\gamma_\ell\right)\sum_{p=0}^{n_\ell}\frac{1}{p!}\left(\frac{m\,\psi_\ell}{T\,\overline{\gamma}_\ell}\gamma_\ell\right)^p\right]. \tag{15}$$

### C. Order Statistics

The instantaneous SNR per symbol at the output of a GSC($K$,3) receiver can be expressed as $\gamma_{\rm gsc} = \sum_{k=1}^{3}\xi_k\,\gamma_{(k)}$, where $\xi_k = 1$, if $k = 1$, 2, $K$, and $\xi_k = 0$, if $k = K+1$, 3, while $\gamma_{(\ell)}$'s are the descending ordered $\gamma_\ell$'s, i.e., $\gamma_{(1)} \geq \gamma_{(2)} \geq \gamma_{(3)}$ (by default, $\xi_1 = 1$). Based on (14) and [22, Appendix], the joint PDF of $\gamma_{(\ell)}$'s can be expressed as

$$f_{\gamma_{(1)},\gamma_{(2)},\gamma_{(3)}}(\gamma_1,\gamma_2,\gamma_3) = \sum_{e_i\in\mathcal{S}_3} f_{\gamma_1,\gamma_2,\gamma_3}\left(\gamma_{e_i[1]},\gamma_{e_i[2]},\gamma_{e_i[3]}\right) \tag{16}$$

with $e_i \in \mathcal{S}_3$ denoting $e_\ell = \{e_i[1], e_i[2], e_i[3]\}$, one specific permutation of the integers $\{1, 2, 3\}$. The MGF of the GSC output SNR per symbol can be obtained from the above equation as $\mathcal{M}_{\gamma_{\rm gsc}}(s) = \mathbb{E}\langle\exp(-s\,\gamma_{\rm gsc})\rangle$. Using an infinite series representation for Bessel functions [30, eq. (8.445)] in (14), this MGF yields

$$\mathcal{M}_{\gamma_{\rm gsc}}(s) = \frac{2^{3\,(m-1)}}{(m-1)\,\left[\det\left(\widetilde{\mathbf{\Sigma}}\right)\right]^m} \times \sum_{k=m-1}^{\infty}\sum_{l_1,l_2,l_3=0}^{\infty} k\,(-1)^{k-m+1}\binom{m+k-2}{2m-3} \times \left[\prod_{\ell=1}^{3}\frac{2^{-k-2l_\ell}\,\widetilde{b}_\ell^{\,2\,l_\ell+k+1-m}}{l_\ell!\,(l_\ell+k)!}\right] \times \sum_{e_i\in\mathcal{S}_3} G\left[s;\,n_{e_i[1]},\widetilde{a}_{e_i[1]};\,n_{e_i[2]},\widetilde{a}_{e_i[2]};\,n_{e_i[3]},\widetilde{a}_{e_i[3]}\right] \tag{17}$$

with $G(s;\,n_1,a_1;\,n_2,a_2;\,n_3,a_3) = \int_0^\infty\int_{\gamma_3}^\infty\int_{\gamma_2}^\infty\{\prod_{i=1}^{3}\gamma_i^{n_i}\exp[-(a_i/2+s\,\xi_i)\,\gamma_i]\}\,{\rm d}\gamma_1\,{\rm d}\gamma_2\,{\rm d}\gamma_3$. Using [30, eqs. (8.350/2) and (8.352/2)], this triple integral can be solved in closed form as

$$G(s;\,n_1,a_1;\,n_2,a_2;\,n_3,a_3) = n_1!\sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2+p_1}(p_1)_{n_2}\,(p_2)_{n_3}\prod_{\ell=1}^{3}\left[(B_\ell+s)\sum_{i=1}^{\ell}\xi_i\right]^{-u_\ell} \tag{18}$$

with $B_\ell = 0.5\sum_{i=1}^{\ell}a_i/\sum_{i=1}^{\ell}\xi_i$, $u_1 = 1+n_1-p_1$, $u_2 = 1+p_1+n_2-p_2$, and $u_3 = n_3+p_2+1$.

Note that for $\xi_1 = \xi_2 = \xi_3 = 1$ ($K = 3$: MRC scheme), (17) numerically agrees with [10, eq. (11)] $\mathcal{M}_{\gamma_{\rm mrc}}(s) = [\det(\mathbf{I}_3 + s\,\widetilde{\mathbf{\Sigma}})]^{-m}$, with $\mathbf{I}_3$ being the $3\times 3$ identity matrix.

### D. Performance Analysis

Based on the above derived expressions, next we analyze the performance of GSC($K$,3) in terms of the ABEP and outage probability.

*1) Error probability:* Using the MGF of triple-branch GSC output SNR per symbol, as given by (17), the ABEP for non-coherent binary frequency shift keying (NBFSK) and binary differential phase shift keying (BDPSK) modulation schemes can be directly calculated (e.g. for BDPSK, $\overline{P}_{be} = 0.5\,\mathcal{M}_{\gamma_{\rm gsc}}(1)$). For other schemes, including binary phase shift keying (BPSK), $M$-ary-phase shift keying ($M$-PSK), quadrature amplitude modulation ($M$-QAM), amplitude modulation ($M$-AM), and differential phase shift keying ($M$-DPSK), single integrals with finite limits and integrands composed of elementary (exponential and trigonometric) functions, have to be readily evaluated via numerical integration [3].

*2) Outage probability:* Let $\gamma_{\rm th}$ be a certain specified outage threshold. Then, the outage probability is defined as the probability that $\gamma_{\rm gsc}$ falls below $\gamma_{\rm th}$, i.e., $P_{\rm out}(\gamma_{\rm th}) = \Pr(\gamma_{\rm gsc} \leq \gamma_{\rm th}) = F_{\gamma_{\rm gsc}}(\gamma_{\rm th})$, where $F_{\gamma_{\rm gsc}}(\cdot)$ is the CDF of $\gamma_{\rm gsc}$. The outage probability can be extracted from $\mathcal{M}_{\gamma_{\rm gsc}}(s)$ based on the following Laplace transformation

$$P_{\rm out}(\gamma_{\rm th}) = \mathbb{L}^{-1}\left\{\frac{\mathcal{M}_{\gamma_{\rm gsc}}(s)}{s};t\right\}\Bigg|_{t=\gamma_{\rm th}}. \tag{19}$$

Two important and generic cases are considered:

- $\widetilde{a}_1 \neq \widetilde{a}_2 \neq \widetilde{a}_3$: By substituting (17) in (19) and due to the linearity of the Laplace transform, we only have to evaluate

$$g(\gamma_{\rm th};\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = \mathbb{L}^{-1}\left\{\frac{1}{s}\, G(s;\, n_1, a_1;\, n_2, a_2;\, n_3, a_3)\,; t\right\}\Big|_{t=\gamma_{\rm th}}. \tag{20}$$

By invoking the theory of rational functions [30, Section 2.102], the above complicated transformation splits to several transformations of the form $\mathbb{L}^{-1}\{(s+B_\ell)^{-q}/s; t\}$ ($q$ integer), which using [30, Section 17.1], can be extracted as $\mathbb{L}^{-1}\{(s+B_\ell)^{-q}/s; t\} = [1 - \exp(-B_\ell\, t)\sum_{p=0}^{q-1}(B_\ell\, t)^p/p!]/B_\ell^q$. Hence, after a lot of algebraic manipulations, an analytical expression for the outage probability of GSC($K$,3) receivers can be obtained from (17), simply replacing $G$'s with

$$g(\gamma_{\rm th};\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = n_1! \sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2+p_1} (p_1)_{n_2}\,(p_2)_{n_3}\left[\prod_{\ell=1}^{3}\left(\sum_{i=1}^{\ell}\xi_i\right)^{-u_\ell}\right] \times \sum_{\ell=1}^{3}\sum_{q=1}^{u_\ell}\frac{\beta_q}{B_\ell^q}\left[1 - \exp(-\gamma_{\rm th}\, B_\ell)\sum_{p=0}^{q-1}\frac{1}{p!}\,(\gamma_{\rm th}\, B_\ell)^p\right] \tag{21}$$

where $\beta_q = \Psi_\ell(s)^{(u_\ell - q)}|_{s=-B_\ell}/(u_\ell - q)!$ and $\Psi_\ell(s) = (s+B_\ell)^{u_\ell}\prod_{n=1}^{3}(B_n + s)^{-u_n}$. This case (with distinct $\widetilde{a}_\ell$'s) is appropriate for studying the linearly arbitrary correlation model with non id channel statistics ($\rho_{1,2} = \rho_{2,3}$ and $\overline{\gamma}_\ell \neq \overline{\gamma}_{\ell'}\ \forall \ell \neq \ell'$).

- $\widetilde{a}_\ell = \widetilde{a}$ ($B_\ell = B$) $\forall \ell$: Similarly to the analysis of the previous case

$$g(\gamma_{\rm th};\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = \frac{n_1!}{B^{3+\sum_{\ell=1}^{3} n_\ell}} \times \left[1 - \exp(-B\,\gamma_{\rm th})\sum_{p=0}^{2+\sum_{\ell=1}^{3} n_\ell}\frac{1}{p!}\,(B\,\gamma_{\rm th})^p\right] \times \sum_{p_1=0}^{n_1}\sum_{p_2=0}^{n_2+p_1}(p_1)_{n_2}\,(p_2)_{n_3}\prod_{\ell=1}^{3}\left(\sum_{i=1}^{\ell}\xi_i\right)^{-u_\ell}. \tag{22}$$

This case (with identical $\widetilde{a}_\ell$'s) is appropriate for studying the constant correlation model with id channel statistics ($\rho_{\ell,\ell'} = \rho\ \forall \ell \neq \ell'$ and $\overline{\gamma}_\ell = \overline{\gamma}$).

Note that for triple-branch SC receivers, a simple outage probability expression can be easily extracted, setting $\gamma_\ell = \gamma_{\rm th}$ $\forall \ell$ in (15), i.e., $P_{\rm out}(\gamma_{\rm th}) = F_{\gamma_1,\gamma_2,\gamma_3}(\gamma_{\rm th}, \gamma_{\rm th}, \gamma_{\rm th})$.

## IV. Numerical and Computer Simulation Results

In this section, in order to provide the applicability and show the usefulness of the proposed analysis, numerical and computer simulation results for GSC($K$,3) receivers operating over arbitrary correlated and not necessarily id Nakagami-$m$ fading channels are provided.

Setting equal summation limits for the truncation of (17) to all sums, Table II summarizes the number of terms needed so as the ABEP of BDPSK to converge with relative error $e_r \leq 5\%$ comparing to accurate computer simulations. The constant correlation model is considered with id channels ($\overline{\gamma}_\ell = \overline{\gamma}$) various values of $\rho$ and $m = 2$, 4. Interestingly enough, only a few terms are required in order the series in (17) to converge. An increase on $\overline{\gamma}$, results to a decrease of the required number of terms, while for a fixed $\overline{\gamma}$, the required number of terms for convergence increases with increasing $m$ and/or $\rho$.

TABLE II
NUMBER OF REQUIRED TERMS FOR CONVERGENCE OF (17) FOR THE CONSTANT CORRELATION MODEL

| | $\rho = 0.1$ | | $\rho = 0.5$ | | $\rho = 0.7$ | |
|---|---|---|---|---|---|---|
| $\overline{\gamma}$ (dB) | $m = 2$ | $m = 4$ | $m = 2$ | $m = 4$ | $m = 2$ | $m = 4$ |
| -5 | 4 | 6 | 7 | 10 | 10 | 19 |
| 0 | 3 | 6 | 6 | 8 | 7 | 13 |
| 5 | 2 | 4 | 4 | 6 | 5 | 8 |
| 10 | 1 | 3 | 3 | 4 | 4 | 7 |

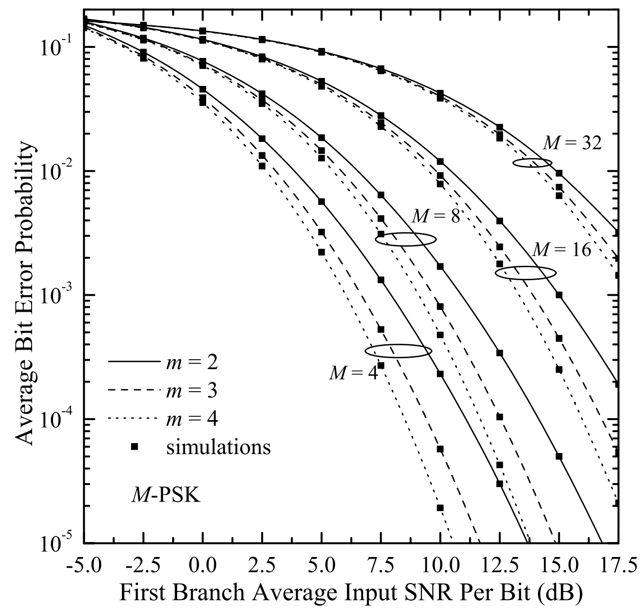


Fig. 1. ABEP of Gray-encoded $M$-PSK with GSC(2,3) receivers for a linearly arbitrary correlation model as a function of the first branch average input SNR per bit.

Next, an exponential power delay profile (PDP) $\overline{\gamma}_\ell = \overline{\gamma}_1\exp[-\delta(\ell-1)]$ with power decaying factor $\delta = 0.1$ is considered. In Fig. 1, the ABEP for Gray-encoded $M$-ary PSK scheme is plotted as a function of the first branch average input SNR per bit $\overline{\gamma}_b = \overline{\gamma}_1/\log_2(M)$. The linearly arbitrary correlation model has been adopted with $\rho_{12} = \rho_{23} = 0.795$ and $\rho_{13} = 0.605$ [10]. As expected, the ABEP improves as $M$ decreases and/or $m$ and/or $\overline{\gamma}_b$ increase. In Figs. 2 and 3, the constant correlation model [3], [9] with $\rho_{\ell\ell'} = 0.1\ \forall \ell \neq \ell'$ has been considered. More specifically, in Fig. 2, a few curves are illustrated for the ABEP of BDPSK modulation as a function of $\overline{\gamma}_b$, several values of $m$, and $K = 1$, 2, 3. Furthermore, in Fig. 3, the outage probability is plotted as a function of the first branch normalized outage threshold, $\overline{\gamma}_1/\gamma_{\rm th}$. For comparison purposes, in Figs. 2 and 3 a few curves for SC and MRC are also included. As expected, it is clear that the MRC scheme outperforms both GSC(2,3) and SC ones.

In all the above figures, i.e., Figs. 1–3, the numerically

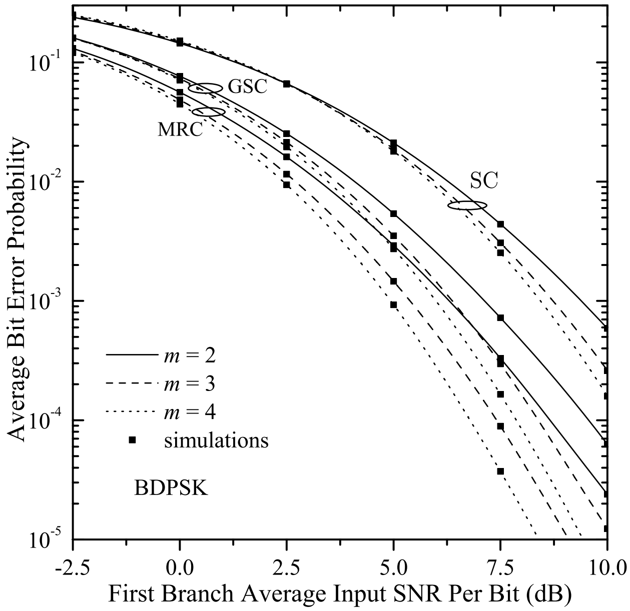


Fig. 2. ABEP of BDPSK with GSC(2,3), MRC, and SC receivers for a constant correlation model as a function of the first branch average input SNR per bit.

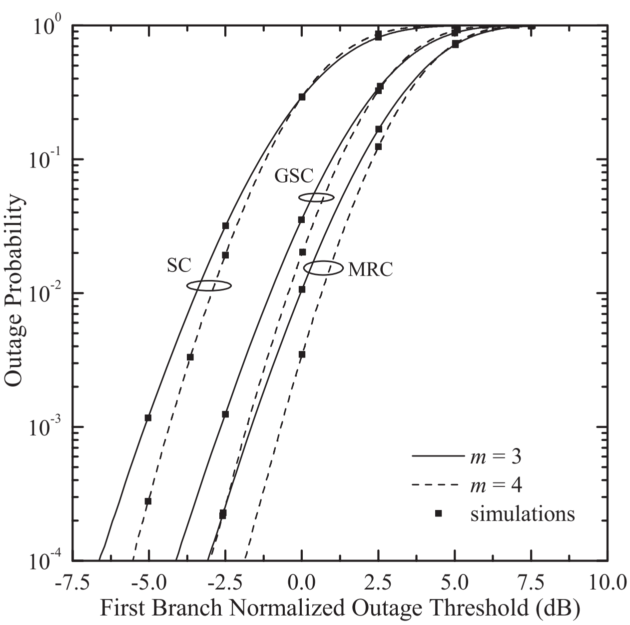


Fig. 3. Outage probability for GSC(2,3), MRC, and SC receivers for a constant correlation model as a function of the first branch normalized outage threshold.

evaluated results are compared to equivalent simulation ones. These comparisons clearly show that the curves for the ABEP coincide with square pattern signs obtained via simulations, verifying the correctness of the proposed analysis.

## V. Conclusions

A rapidly convergent infinite series representation of a trivariate Nakagami-$m$ PDF with arbitrary covariance matrix was derived from the diagonal elements of the Wishart matrix. Following the MGF-based approach and by extracting the MGF of the GSC output SNR, the performance of GSC receivers was analyzed and compared to conventional ones such as MRC and SC. Finally, extensive numerical and computer simulation results were presented and compared, and a perfect match was observed.

## References


[1] M. Nakagami, "The $m$-distribution–a general formula of intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*. Oxford, U.K.: Pergamon Press, 1960, pp. 3–36.

[2] A. U. Sheikh, M. Abdi, and M. Handforth, "Indoor mobile radio channel at 946 MHz: measurements and modeling," in *Proc. IEEE Veh. Technol. Conf.*, Secaucus, NJ, May 1993, pp. 73–76.

[3] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd ed. New York: Wiley, 2005.

[4] C. C. Tan and N. C. Beaulieu, "Infinite series representation of the bivariate Rayleigh and Nakagami-$m$ distributions," *IEEE Trans. Commun.*, vol. 45, no. 10, pp. 1159–1161, Oct. 1997.

[5] J. Reig, L. Rubio, and N. Cardona, "Bivariate Nakagami-$m$ distribution with arbitrary fading parameters," *Electron. Lett.*, vol. 38, no. 25, pp. 1715–1717, Dec. 2002.

[6] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "On the multivariate Nakagami-$m$ distribution with exponential correlation," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1240–1244, Aug. 2003.

[7] ——, "An effiicient approach to multivariate Nakagami-$m$ distribution using Green's matrix approximation," *IEEE Trans. Wireless Commun.*, vol. 2, no. 5, pp. 883–889, Sept. 2003.

[8] ——, "Performance analysis of triple selection diversity over exponentially correlated Nakagami-$m$ fading channels," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1245–1248, Aug. 2003.

[9] V. A. Aalo, "Performance of maximal-ratio diversity systems in a correlated Nakagami-fading environment," *IEEE Trans. Commun.*, vol. 43, no. 8, pp. 2360–2369, Aug. 1995.

[10] Q. T. Zhang, "Maximal-ratio combining over Nakagami fading channels with an arbitrary branch covariance matrix," *IEEE Trans. Veh. Technol.*, vol. 48, no. 4, pp. 1141–1150, July 1999.

[11] M.-S. Alouini, A. Abdi, and M. Kaveh, "Sum of gamma variates and performance of wireless communication systems over Nakagami-fading channels," *IEEE Trans. Veh. Technol.*, vol. 50, no. 6, pp. 1471–1480, Nov. 2001.

[12] J. Luo, J. R. Zeidler, and S. McLaughlin, "Performance analysis of compact antenna arrays with MRC in correlated Nakagami fading channels," *IEEE Trans. Veh. Technol.*, vol. 50, no. 1, pp. 267–277, Jan. 2001.

[13] M. Z. Win, G. Chrisikos, and J. H. Winters, "MRC performance for $M$-ary modulation in arbitrarily correlated Nakagami fading channels," *IEEE Commun. Lett.*, vol. 4, no. 10, pp. 301–303, Oct. 2000.

[14] Q. T. Zhang and H. G. Lu, "A general analytical approach to multi-branch selection combining over various spatially correlated fading channels," *IEEE Trans. Commun.*, vol. 50, no. 7, pp. 1066–1073, July 2002.

[15] V. A. Aalo, T. Piboongungon, and G. P. Efthymoglou, "Another look at the performance of MRC schemes in Nakagami-$m$ fading channels with arbitrary parameters," *IEEE Trans. Commun.*, vol. 53, no. 12, pp. 2002–2005, Dec. 2005.

[16] R. K. Mallik and M. Z. Win, "Analysis of hybrid selection/maximal-ratio combining in correlated Nakagami fading," *IEEE Trans. Commun.*, vol. 50, no. 8, pp. 1372–1383, Aug. 2002.

[17] R. Kwan, P. Ho, and C. Leung, "On joint order statistics in correlated Nakagami fading channels," in *Proc. IEEE Wirel. Commun. Networking Conf.*, Kowloon, China, Mar. 2007, pp. 2043–2047.

[18] N. C. Sagias and K. Peppas, "Error rate analysis of threshold-based hybrid selection/maximal-ratio diversity over correlated Nakagami-$m$ fading channels," *IEEE Commun. Lett.*, vol. 11, no. 12, pp. 922–924, Dec. 2007.

[19] X. Zhang and N. C. Beaulieu, "Performance analysis of generalized selection combining in generalized correlated Nakagami-$m$ fading," *IEEE Trans. Commun.*, vol. 54, no. 11, pp. 2103–2112, Nov. 2006.

[20] ——, "SER of threshold-based hybrid selection/maximal-ratio combining in correlated Nakagami fading," *IEEE Trans. Commun.*, vol. 53, no. 9, pp. 1423–1426, Sept. 2005.

[21] ——, "SER of threshold-based hybrid selection/maximal-ratio combining in equicorrelated Nakagami fading," *IEEE Commun. Lett.*, vol. 8, no. 9, pp. 552–554, Aug. 2005.

[22] ——, "Approximate SER of H-S/MRC in Nakagami fading with arbitrary branch power correlation," *IEEE Trans. Commun.*, vol. 54, no. 1, pp. 56–60, Jan. 2006.

[23] D. R. Jensen, "The joint distribution of traces of Wishart matrices and some applications," *Ann. Math. Stat*, vol. 41, no. 1, pp. 133–145, Feb. 1970.

[24] P. R. Krishnaiah, P. Hagis, and L. Steinberg, "A note on the bivariate chi distribution," *SIAM Review*, vol. 5, no. 2, pp. 140–144, Apr. 1963.

[25] A. S. Krishnamoorthy and P. Parthasarathy, "A multivariate gamma-type distribution," *Ann. Math. Stat.*, vol. 22, no. 4, pp. 549–557, Dec. 1951.

[26] K. S. Miller, *Multidimensional Gaussian distributions*. New York: Wiley, 1964.

[27] K. S. Miller, R. I. Bernstein, and L. E. Blumenson, "Generalized Rayleigh processes," *Quart. Appl. Math*, vol. 16, pp. 137–145, 1958.

[28] T. Royen, "Expansions for the multivariate chi-square distribution," *J. Multivariate Anal.*, vol. 38, no. 2, pp. 213–232, Aug. 1991.

[29] M. Hagedorn, P. J. Smith, P. J. Bones, R. P. Millane, and D. Pairman, "A trivariate chi-square distribution derived from the complex Wishart distribution," *J. Multivariate Anal.*, vol. 97, no. 3, pp. 655–674, Mar. 2006.

[30] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 6th ed. New York: Academic Press, 2000.

[31] Y. Chen and C. Tellambura, "Infinite series representations of the trivariate and quadrivariate Rayleigh distribution and their applications," *IEEE Trans. Commun.*, vol. 53, no. 12, pp. 2092–2101, Dec. 2005.

[32] P. Dharmawansa, N. Rajatheva, and C. Tellambura, "Infinite series representations of the trivariate and quadrivariate Nakagami-$m$ distributions," *IEEE Trans. Wireless Commun.*, vol. 6, no. 12, pp. 4320–4328, Dec. 2007.