On the Multivariate Weibull Fading Model With Arbitrary Correlation Matrix

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***Abstract*—An efficient approximation for the multivariate Weibull distribution with arbitrary correlations is presented. By approximating the Gaussian correlation matrix with a Green's matrix, a useful analytical expression for the joint distribution of the fading envelopes is derived. As an application, the outage performance of multibranch receivers operating over arbitrarily correlated Weibull fading channels is studied. Numerical and computer simulation results are presented and compared to illustrate the accuracy of the proposed approximation.**

***Index Terms*—Correlated fading, diversity, Green's matrix, multichannel reception, multivariate analysis, outage probability, Rayleigh fading, selection combining (SC), Weibull fading.**

I. Introduction

MULTIVARIATE statistics provides useful mathematical tools for modeling and analyzing realistic wireless propagation channels with correlated fading [1]. Such fading channels are usually met in contemporary digital communication systems employing multibranch receivers with insufficiently separated antennas [2], where space or polarization diversity is applied, e.g., handheld mobile terminals and indoor base stations.

In an early work regarding Gaussian class multivariate distributions [3], the generalized Rayleigh distribution has been studied for the special case in which the correlation matrix has a specific form so that its inverse matrix is tridiagonal. In a later work, exact joint cumulative distribution function (cdf) and probability density function (pdf) expressions, for three and four Rayleigh random variables (RVs), have been derived [4]. An efficient approach for the evaluation of the multivariate Rayleigh and Nakagami-m cdf and pdf has been presented in [5], which extends the results presented in [3], based on an approximation for the correlation matrix with a Green's matrix [6]. This approximation has been also used in [7], where the performance of a threshold-based hybrid selection/maximal-ratio combining (T-HC/MRC) receiver has been analyzed. Another useful distribution for fading channel modeling is Weibull, which exhibits good fit for both indoor and outdoor environments [8]. In [9], the outage probability (OP) of a triple-branch selection combining (SC) receiver over arbitrarily correlated Weibull fading has been studied. However, to the best of the authors' knowledge, for more than three correlated branches, there is not any publication addressing the multivariate Weibull distribution with an arbitrary correlation matrix.

In this letter, by approximating the Gaussian correlation matrix with a Green's matrix, we extend [8] and [9], deriving an analytical approximation for the multivariate Weibull cdf for arbitrary correlations. The usefulness of the proposed analysis is outlined through comparisons with extensive computer simulations, for the OP of multibranch SC receivers.

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II. Weibull Statistics and Approximations

Let $\mathbf{Y}_k = [Y_{k,1}\, Y_{k,2} \cdots Y_{k,L}]^\dagger (k = 1,2)$ be two L-dimensional real column vectors ($\dagger$ denotes the transpose), which are independent and identically distributed zero mean $\mathbb{E}\langle Y_{k,l}\rangle = 0$ with variance $\mathbb{E}\langle Y_{k,l}^2\rangle = \sigma^2$ ($l = 1,2,\ldots,L$ and $\mathbb{E}\langle\cdot\rangle$ denotes expectation) Gaussian RVs having a symmetric and positive–definite correlation matrix $\mathbf{\Sigma}_g \in \Re^{L\times L}$. Also, let $R_l = \sqrt{Y_{1,l}^2 + Y_{2,l}^2}$. Then, R_ls are correlated Rayleigh RVs with power correlation matrix $\mathbf{\Sigma}_r$ (between R_i^2 and $R_j^2, i,j = 1,2,\ldots,L$), and marginal cdfs given by $F_{R_l}(r) = 1 - \exp(-r^2/\Omega)$, where $\Omega = \mathbb{E}\langle R_l^2\rangle = 2\sigma^2$ is the corresponding average power. The correlation matrix of the underlying Gaussian processes $\mathbf{\Sigma}_g$ is related to $\mathbf{\Sigma}_r$ as $\mathbf{\Sigma}_g = \sqrt{\mathbf{\Sigma}_r}$ ($\sqrt{\mathbf{\Sigma}_r}$ stands for a matrix with elements the square root ones of $\mathbf{\Sigma}_r$). When $\boldsymbol{W} = \mathbf{\Sigma}_g^{-1}$ is tridiagonal ($\mathbf{\Sigma}_g^{-1}$ stands for the inverse of $\mathbf{\Sigma}_g$), according to [3, Th. I], the joint cdf of $\mathbf{R} = [R_1\, R_2 \cdots R_L]$ is given by (1), as shown at the bottom of the next page, with $|\boldsymbol{W}|$ being the determinant of $\boldsymbol{W}, p_{i,j} = \boldsymbol{W}_{i,j} \in \Re$, and $\gamma(\cdot,\cdot)$ being the lower incomplete Gamma function [10, eq. (8.350/1)].

A. Multivariate Weibull Distribution

To model the multivariate Weibull distribution with arbitrary correlation, we introduce $\beta > 0$ as the fading parameter. Applying a power transformation

$$Z_l = R_l^{2/\beta} \tag{2}$$

the joint Weibull cdf of $\mathbf{Z} = [Z_1\, Z_2 \cdots Z_L]$ is obtained as

$$F_{\mathbf{Z}}(z_1, z_2, \ldots, z_L) = F_{\mathbf{R}}\left(z_1^{\beta/2}, z_2^{\beta/2}, \ldots, z_L^{\beta/2}\right) \tag{3}$$

having Weibull marginal cdfs given by

$$F_{Z_l}(z) = 1 - \exp\left(-\frac{z^\beta}{\Omega}\right) \tag{4}$$

with $\Omega = \mathbb{E}\langle Z_l^{\beta}\rangle$. As β increases, the severity of the fading decreases, while for $\beta = 2$, (3) reduces to (1). The (Weibull) power correlation matrix $\mathbf{\Sigma} \in \Re^{L\times L}$ is given by $\mathbf{\Sigma}_{i,j} \equiv 1$ for $i = j$ and $\mathbf{\Sigma}_{i,j} = \mathbf{\Sigma}_{j,i} \equiv \rho_{i,j}$ for $i \neq j$, with $0 \leq \rho_{i,j} < 1$ being the power correlation coefficient between Z_i^2 and Z_j^2 [1, eq. (9.195)]. The Rayleigh ($\mathbf{\Sigma}_r$) and the Weibull ($\mathbf{\Sigma}$) power correlation matrices are related based on [8, eq. (16)]. Note also that taking the Lth-order mixed derivatives of (3), the joint pdf of $\mathbf{Z}$ can be easily obtained.

B. Green's Matrix Approximation

As we have already mentioned, in order for (1) and (3) to hold, $\boldsymbol{W}$ must have a tridiagonal structure. Generally, $\boldsymbol{W}$ may not be tridiagonal as in the case of arbitrary correlation. In such cases, an efficient approximation to $\mathbf{\Sigma}_g$ may be applied. Specifically, the correlation matrix $\mathbf{\Sigma}_g$ can be approximated by a Green's matrix $\boldsymbol{C}$, with its elements being the closest possible values to the ones of $\mathbf{\Sigma}_g$. Note that the inverse matrix of a Green's matrix is always tridiagonal [6]. By approximating $\mathbf{\Sigma}_g$ with $\boldsymbol{C}$, a nonlinear system of equations is formed. This system can be solved using local minimization methods such as Levenberg–Marquardt or quasi-Newton [11], available in most of the well-known mathematical software packages, e.g., MATHEMATICA. By replacing $\boldsymbol{W}$ with $\boldsymbol{C}^{-1}$ in (3) (also $p_{i,j}$ with $\boldsymbol{C}^{-1}_{i,j}$), an approximate multivariate Weibull joint cdf with arbitrary correlation can be obtained.

C. Outage Probability of Multibranch SC Receivers

Let a signal's transmission over the lth flat Weibull fading channel corrupted with additive white Gaussian noise (AWGN), with E_s being the symbols' energy and N_0 the single-sided noise power spectral density of the AWGN. The instantaneous signal-to-noise ratio (SNR) of the lth diversity channel can be expressed by $\gamma_l = Z_l^2\, E_s/N_0$ with its corresponding average SNR being $\bar{\gamma}_l = \mathbb{E}\langle Z_l^2\rangle E_s/N_0 = \Gamma(1+2/\beta)\Omega^{2/\beta}E_s/N_0 = \bar{\gamma}\,\forall\, l$, where $\Gamma(\cdot)$ is the Gamma function [10, eq. (8.310/1)]. Based on an interesting property of the Weibull distribution, that the nth power of a Weibull distributed RV with parameters (β,Ω) is another Weibull distributed RV with parameters $(\beta/n,\Omega)$, it can be easily concluded that γ_l is also a Weibull distributed RV with parameters $(\beta/2,(\alpha\bar{\gamma})^{\beta/2})$ and $\alpha = 1/\Gamma(1+2/\beta)$. Hence, (3) can be used to derive the joint cdf of $\boldsymbol{\gamma} = [\gamma_1\,\gamma_2\,\cdots\,\gamma_L]$.

The instantaneous SNR at the output of an SC receiver will be the one with the highest instantaneous value among the L branches, i.e., $\gamma_{\rm sc} = \max\{\gamma_1,\gamma_2,\ldots,\gamma_L\}$. The OP, $P_{\rm out}$, is defined as the probability that the SC output SNR falls below a given outage threshold $\gamma_{\rm th}$. This probability can be easily obtained as $P_{\rm out}(\gamma_{\rm th}) = F_{\boldsymbol{\gamma}}(\gamma_{\rm th},\gamma_{\rm th},\ldots,\gamma_{\rm th})$, resulting in (5), as shown at the bottom of the page.

TABLE I
NUMBER OF TERMS FOR CONVERGENCE TO THE SIXTH SIGNIFICANT DIGIT FOR THE OP OF SC WITH A LINEARLY ARBITRARY MODEL ($L = 3$)

$\gamma_{\rm th}/\bar{\gamma}$ (dB)	$\beta = 1.5$	$\beta = 2.0$	$\beta = 2.5$	$\beta = 3.7$	$\beta = 4.5$
-20	1	1	1	1	1
-15	3	2	1	1	1
-10	3	3	2	2	1
-5	4	3	3	2	2
0	7	7	6	6	6

III. NUMERICAL AND COMPUTER SIMULATION RESULTS

In this section, in order to show the usefulness and to examine the accuracy of the previous analysis, numerical and computer simulation results are presented for a multibranch SC receiver operating over arbitrary correlated Weibull fading channels. The SC receiver employs linear arrays, with the antennas not to be placed unevenly, which is known as linearly arbitrary correlation model [2]. These results are compared with extensive computer simulations ones, generated from correlated Rayleigh RVs after applying (2).

Equation (5) requires the summation of an infinite number of terms. Table I summarizes the number of terms in each sum needed, so (5) converges after the truncation of the infinite series for a linearly arbitrary correlation model with $L = 3$ given in [5, p. 887]. As Table I indicates, an increase in β results in a decrease of the required number of terms that are essential to be summed in order $P_{\rm out}$ to converge. Moreover, the number of the required terms depends strongly on the normalized outage threshold

$$F_{\mathbf{R}}(r_1,r_2,\ldots,r_L) = |\boldsymbol{W}| \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \frac{\prod_{i=1}^{L-1}\left(p_{i,i+1}^{k_i}\big/ k_i!\right)^2}{p_{1,1}^{k_1+1}\left(\prod_{i=2}^{L-1} p_{i,i}^{k_{i-1}+k_i+1}\right)p_{L,L}^{k_{L-1}+1}} \times\gamma\left(k_1+1,\frac{p_{1,1}}{\Omega}r_1^2\right)\gamma\left(k_{L-1}+1,\frac{p_{L,L}}{\Omega}r_L^2\right)\prod_{i=2}^{L-1}\gamma\left(k_{i-1}+k_i+1,\frac{p_{i,i}}{\Omega}r_i^2\right). \quad (1)$$

$$P_{\rm out}(\gamma_{\rm th}) = |\boldsymbol{W}| \sum_{k_1,k_2,\ldots,k_{L-1}=0}^{\infty} \frac{\prod_{i=1}^{L-1}\left(p_{i,i+1}^{k_i}\big/ k_i!\right)^2}{p_{1,1}^{k_1+1}\left(\prod_{i=2}^{L-1} p_{i,i}^{k_{i-1}+k_i+1}\right)p_{L,L}^{k_{L-1}+1}}\,\gamma\left[k_1+1,p_{1,1}\left(\frac{\gamma_{\rm th}}{\alpha\,\bar{\gamma}}\right)^{\beta/2}\right] \times\gamma\left[k_{L-1}+1,p_{L,L}\left(\frac{\gamma_{\rm th}}{\alpha\,\bar{\gamma}}\right)^{\beta/2}\right]\prod_{i=2}^{L-1}\gamma\left[k_{i-1}+k_i+1,p_{i,i}\left(\frac{\gamma_{\rm th}}{\alpha\,\bar{\gamma}}\right)^{\beta/2}\right]. \quad (5)$$

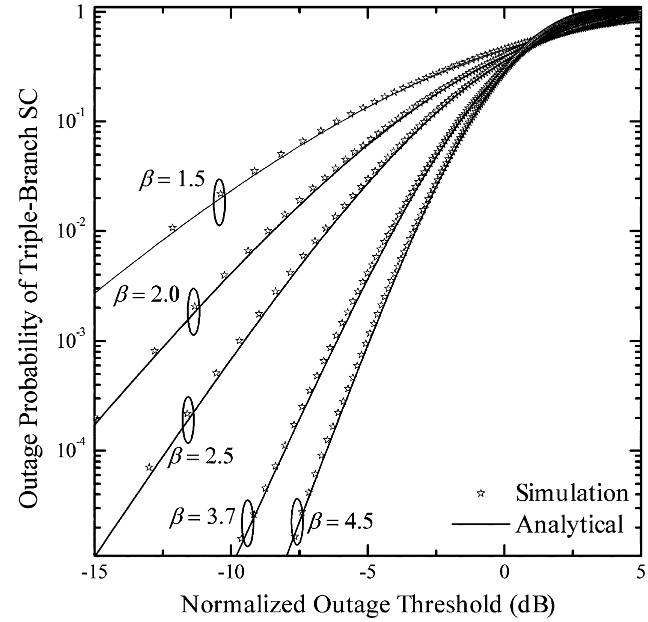

Fig. 1. OP of a triple-branch SC receiver as function of the normalized outage threshold for a linearly arbitrary correlation model.

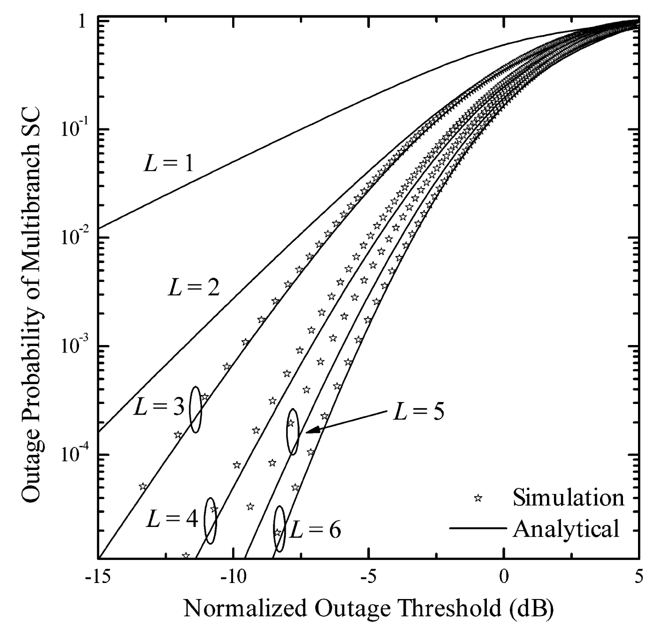

Fig. 2. OP of a multibranch SC receiver as function of the normalized outage threshold for $\beta = 2.5$ and linearly arbitrary correlation models.

$\gamma_{\rm th}/\bar{\gamma}$. As $\gamma_{\rm th}/\bar{\gamma}$ decreases, fewer terms are required. Also, it is interesting to mention that from additional comparisons that were conducted, the convergence rate does not depend on L.

Having numerically evaluated (5), in Fig. 1, $P_{\rm out}$ is plotted as a function of the $\gamma_{\rm th}/\bar{\gamma}$, for a triple-branch SC receiver, with different values of β and a linearly arbitrary correlation matrix given in [5, p. 886]. It can be easily verified that $P_{\rm out}$ degrades with a decrease on β and/or an increase on $\gamma_{\rm th}/\bar{\gamma}$. More importantly, the obtained results clearly show that the approximate curves for $P_{\rm out}$ using Green's matrix are sufficiently close to their corresponding simulation results for different values of β. In Fig. 2, $P_{\rm out}$ is plotted as a function of the $\gamma_{\rm th}/\bar{\gamma}$, for multi-branch SC receivers, with $\beta = 2.5$ and with a linearly arbitrary correlation matrix for $L = 3$ given in [5, p. 886], for $L = 4$ given in [5, p. 887], for $L = 5$ given in [2, eq. (40)], and for $L = 6$ given by

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.629 & 0.363 & 0.200 & 0.139 & 0.079 \\ 0.629 & 1 & 0.629 & 0.363 & 0.200 & 0.139 \\ 0.363 & 0.629 & 1 & 0.629 & 0.363 & 0.200 \\ 0.200 & 0.363 & 0.629 & 1 & 0.629 & 0.363 \\ 0.139 & 0.200 & 0.363 & 0.629 & 1 & 0.629 \\ 0.079 & 0.139 & 0.200 & 0.363 & 0.629 & 1 \end{bmatrix}.$$

In Fig. 2 and for comparison purposes, $P_{\rm out}$ curves for $L = 1$ and $L = 2$ with $\rho_{1,2} = 0.3$ are also included. As expected, $P_{\rm out}$ significantly improves as L increases. Note once again that as this figure indicates, the approximate curves for $P_{\rm out}$ using Green's matrices are very close to their corresponding simulation results for different values of L.

IV. Conclusion

An efficient approximation for the evaluation of the multivariate Weibull distribution with arbitrary correlation was presented. By approximating the Gaussian correlation matrix with a Green's matrix, an accurate analytical formula for the multivariate Weibull cdf was derived. Comparisons between numerically evaluated and extensive computer simulation results verified the accuracy of the proposed approximation for the OP of multibranch SC receivers. The proposed approximation may also be applied to other multivariate distributions.

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