# A Closed-Form Upper-Bound for the Distribution of the Weighted Sum of Rayleigh Variates

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***Abstract*— The problem of finding the distribution of the sum of more than two Rayleigh fading envelopes has never been solved in terms of tabulated functions. In this letter, we present a closed-form union upper-bound for the cumulative distribution function of the weighted sum of $N$ independent Rayleigh fading envelopes. Computer simulation results verify the tightness of the proposed bound for several values of $N$. The proposed bound can be efficiently applied in various wireless applications, such as satellite communications, equal-gain receivers, and radars.**

***Index Terms*— Equal-gain diversity, false alarm probability, Rayleigh fading, sum of fading envelopes.**

## I. INTRODUCTION

THE theoretical analysis of wireless digital communications systems usually deals with complicated and cumbersome statistical tasks, where an analytical solution is very difficult, if not impossible, to be extracted in terms of tabulated functions. The calculation of the cumulative distribution function (CDF) of the weighted sum of $N$ statistically independent Rayleigh fading envelopes is one of them, which arises in several wireless applications. For example, the distribution of such a sum is required for the calculation of the error bounds for coding on generalized mobile satellite fading channels [1]. Another important application is related to the performance analysis of equal-gain combining receivers, in which the received faded signals are equally weighted, cophased, and summed to produce the output signal. Furthermore, in the scientific field of radar receivers, the decision level for a preassigned false-alarm probability requires determining of the CDF of such sums [2], while they can be useful in other important applications which are related to signal detection and linear equalizing, as well as intersymbol interference and phase jitter analysis.

Despite the usefulness of the CDF of the sum of $N$ Rayleigh distributed random variables (RV)s, a closed-form solution has not been given for more than 90 years when $N > 2$. Several attempts to address this problem using approximative solutions have been presented by several authors. In a pioneer work of Beaulieu [3], an infinite series approach for determining this CDF has been developed. This paper also lists all the related works on this topic up to that time. In two other papers, Helstrom has computed the distribution of the sum using saddle-point integration for uniformly weighted RVs [4], as well as for arbitrary weights [5]. More recently, Karagiannidis and Kotsopoulos have presented a semi-analytical approach based on Hermite numerical integration, for the calculation of the CDF of the weighted sum of Nakagami-$m$ and Ricean RVs [6]. However, although the problem of finding the distribution of the sum of Rayleigh distributed RVs has been well-studied, all presented methods are approximative solutions in which the truncation error has to be taken into account. The use of bounds, as opposed to approximations, serves as a safe technique of addressing this problem in a computational efficient and easy way.

In this letter, a closed-form solution for the distribution of the product of $N$ independent Rayleigh distributed RVs is presented. Using the well-known inequality between arithmetic and geometric mean, also applied in [7] for the sum of Lognormal variates, an efficient closed-form union upper-bound for the CDF of the weighted sum of $N$ independent Rayleigh distributed RVs is derived. The tightness of the proposed bound is verified by comparison with performance evaluation results of the exact CDF, obtained by means of computer simulations.

## II. AN UPPER-BOUND FOR THE DISTRIBUTION OF THE WEIGHTED SUM OF RVS

Let $\{X_\ell\}_{\ell=1}^{N}$ be $N$ statistically independent and identically distributed Rayleigh RVs having the probability density function (PDF)

$$f_{X_\ell}(x) = \frac{2\,x}{\Omega}\exp\left(-\frac{x^2}{\Omega}\right) \qquad (1)$$

where $x \geq 0$ and $\Omega = \mathcal{E}\left\langle X_\ell^2\right\rangle\ \forall \ell$, with $\mathcal{E}\left\langle\cdot\right\rangle$ denoting statistical averaging.

We define a new RV $X$, as the product of $N$ RVs $X_\ell$, i.e.,

$$X \triangleq \prod_{i=1}^{N} X_i. \qquad (2)$$

Using the well-known inequality between arithmetic and geometric mean [8, Section 11.116]

$$\mathcal{A}_N \geq \mathcal{G}_N \qquad (3)$$

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for the weighted RVs $k_\ell X_\ell$, where $k_\ell$'s are $N$ positive constant weights, $\mathcal{G}_N$ is the geometric mean

$$\mathcal{G}_N \triangleq \prod_{i=1}^{N} (k_i X_i)^{1/N} \tag{4}$$

and $\mathcal{A}_N$ is the arithmetic mean

$$\mathcal{A}_N \triangleq \frac{1}{N} \sum_{i=1}^{N} k_i X_i \tag{5}$$

a lower-bound for RV $S$, defined as the weighted sum of $X_\ell$'s, i.e.,

$$S \triangleq \sum_{i=1}^{N} k_i X_i \tag{6}$$

can be obtained as

$$S \geq X^{1/N} \left( N \prod_{i=1}^{N} k_i^{1/N} \right). \tag{7}$$

In order to study the statistics of $S$, an expression for the distribution of $X$ is needed to be derived. The moment-generating function (MGF) of $X$ is

$$\begin{aligned} \mathcal{M}_X(s) &= \mathcal{E}\left\langle \exp(-s\,X) \right\rangle \\ &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \exp\left(-s \prod_{i=1}^{N} x_i\right) \prod_{i=1}^{N} f_{X_i}(x_i)\, dx_1\, dx_2 \cdots dx_N \end{aligned} \tag{8}$$

which using (1) can be written as

$$\begin{aligned} \mathcal{M}_X(s) = \frac{2^N}{\Omega^N} & \\ \times \int_0^\infty & x_N \exp\left(-\frac{x_N^2}{\Omega}\right) \cdots \int_0^\infty x_2 \exp\left(-\frac{x_2^2}{\Omega}\right) \\ \times \int_0^\infty & x_1 \exp\left(-s \prod_{i=1}^{N} x_i\right) \exp\left(-\frac{x_1^2}{\Omega}\right) dx_1\, dx_2 \cdots dx_N. \end{aligned} \tag{9}$$

The first integral in (9), i.e., the one on $x_1$, is of the form

$$\mathcal{I}_1 = \int_0^\infty x_1 \exp(-s\,W_2\,x_1) \exp\left(-\frac{x_1^2}{\Omega}\right) dx_1 \tag{10}$$

where $W_\ell = \prod_{i=\ell}^{N} x_i$. Integrals of the form of (10) can be solved using [8, eq. (3.462/5)] and the solution can be transformed [9, eq. (12)] in terms of the tabulated Meijer's G-function [8, eq. (9.301)], leading to

$$\mathcal{I}_1 = \frac{2}{\sqrt{\pi}\,(s\,W_2)^2}\, G_{2,1}^{1,2} \left[ \frac{4}{\Omega\,(sW_2)^2} \,\middle|\, \begin{matrix} -1/2\,,\ 0 \\ 0 \end{matrix} \right]. \tag{11}$$

Using the above solution for $\mathcal{I}_1$, the second integral in (9), i.e., the one on $x_2$, can be written as

$$\begin{aligned} \mathcal{I}_2 = \frac{2}{\sqrt{\pi}\,(s\,W_3)^2} \int_0^\infty & x_2^{-1}\, G_{0,1}^{1,0} \left[ \frac{x_2^2}{\Omega} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right] \\ & \times G_{2,1}^{1,2} \left[ \frac{4/x_2^2}{\Omega\,(s\,W_3)^2} \,\middle|\, \begin{matrix} -1/2\,,\ 0 \\ 0 \end{matrix} \right] dx_2 \end{aligned} \tag{12}$$

which after applying the transformation $y = x_2^2$ and by using [8, eq. (9.31.2)], yields

$$\begin{aligned} \mathcal{I}_2 = \frac{1}{\sqrt{\pi}\,(s\,W_3)^2} \int_0^\infty & y^{-1}\, G_{0,1}^{1,0} \left[ \frac{y}{\Omega} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right] \\ & \times G_{1,2}^{2,1} \left[ \frac{(s\,W_3)^2\, y}{4/\Omega} \,\middle|\, \begin{matrix} 1 \\ 3/2\,,\ 1 \end{matrix} \right] dy. \end{aligned} \tag{13}$$

With the aid of [9, eq. (21)], the above integral can be solved as

$$\mathcal{I}_2 = \frac{1}{\sqrt{\pi}\,(s\,W_3)^2}\, G_{2,2}^{2,2} \left[ \frac{(s\,W_3)^2}{4}\,\Omega^2 \,\middle|\, \begin{matrix} 1\,,\ 1 \\ 3/2\,,\ 1 \end{matrix} \right]. \tag{14}$$

Following the same procedure, the $N$-fold integral in (9), can be expressed in closed-form as

$$\mathcal{M}_X(s) = \frac{4\,s^{-2}}{\sqrt{\pi}\,\Omega^N}\, G_{N,2}^{2,N} \left[ \frac{\Omega^N}{4}\, s^2 \,\middle|\, \begin{matrix} 1, 1, \ldots, 1 \\ 3/2\,,\ 1 \end{matrix} \right]. \tag{15}$$

Taking into account that the PDF of $X$ can be derived from its MGF as

$$f_X(x) = \mathcal{L}^{-1}\left\{ \mathcal{M}_X(s); x \right\} \tag{16}$$

where $\mathcal{L}^{-1}(\cdot;\cdot)$ denotes Laplace transform inversion, using [10] we obtain

$$f_X(x) = \frac{2\,x}{\Omega^N}\, G_{0,N}^{N,0} \left[ \frac{x^2}{\Omega^N} \,\middle|\, \begin{matrix} - \\ 0, 0, \ldots, 0 \end{matrix} \right]. \tag{17}$$

Note, that for $N = 2$ and by using [9, eq. (14)], (17) reduces to a previously known result [11, eq. (90)] given by Nakagami

$$f_X(x) = \frac{4\,x}{\Omega^2}\, K_0\left(\frac{2\,x}{\Omega}\right) \tag{18}$$

where $K_0(\cdot)$ is the zeroth order modified Bessel function of the second kind [8, Section 8.40]. Using (17) and [9, eq. (26)], the CDF of the product of $N$ Rayleigh RVs can be expressed in closed-form as

$$F_X(x) = \frac{x^2}{\Omega^N}\, G_{1,N+1}^{N,1} \left[ \frac{x^2}{\Omega^N} \,\middle|\, \begin{matrix} 0 \\ 0, 0, \ldots, 0, -1 \end{matrix} \right]. \tag{19}$$

Having a readily available expression for the CDF of $X$ as shown in (19) and from (7), an upper-bound for the CDF of the sum of $N$ weighted independent Rayleigh RVs can be obtained in closed-form as

$$\begin{aligned} F_S(x) \leq & \frac{(x/N)^{2N}}{\Omega^N \prod_{i=1}^{N} k_i^2} \\ & \times G_{1,N+1}^{N,1} \left[ \frac{(x/N)^{2N}}{\Omega^N \prod_{i=1}^{N} k_i^2} \,\middle|\, \begin{matrix} 0 \\ 0, 0, \ldots, 0, -1 \end{matrix} \right]. \end{aligned} \tag{20}$$

We have to mention, that the above form of Meijer's G-function can be written in terms of more familiar generalized Hypergeometric [8, eq. (9.14/1)] functions, but it is not presented due to space limitations. Moreover, both Meijer's and generalized Hypergeometric functions are included as built in functions in most of popular mathematical software packages, which are useful for numerical evaluation. Note that alternatively, the problem of finding the CDF of $S$ may be equally stated as of finding the sum of $N$ non-identically distributed (but equally-weighted) RVs with $k_\ell^2\,\Omega$ average power each.

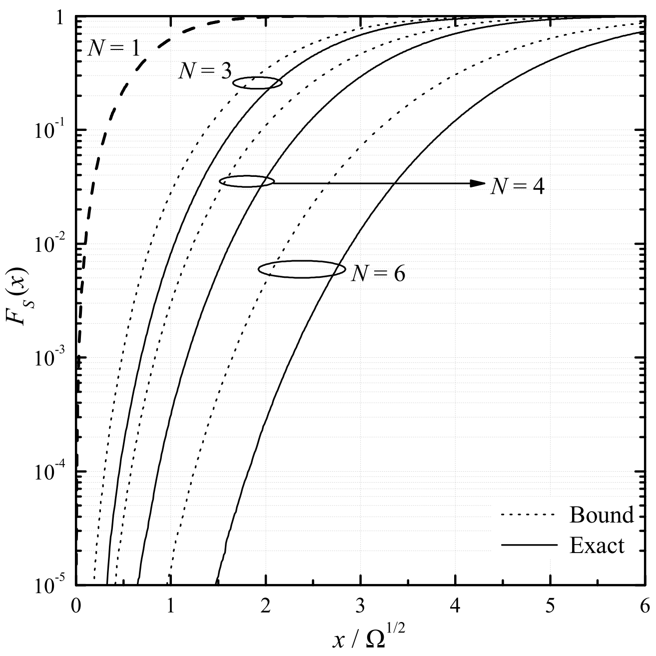


Fig. 1. Bounded and exact CDF of the sum of equally weighted Rayleigh distributed RVs.

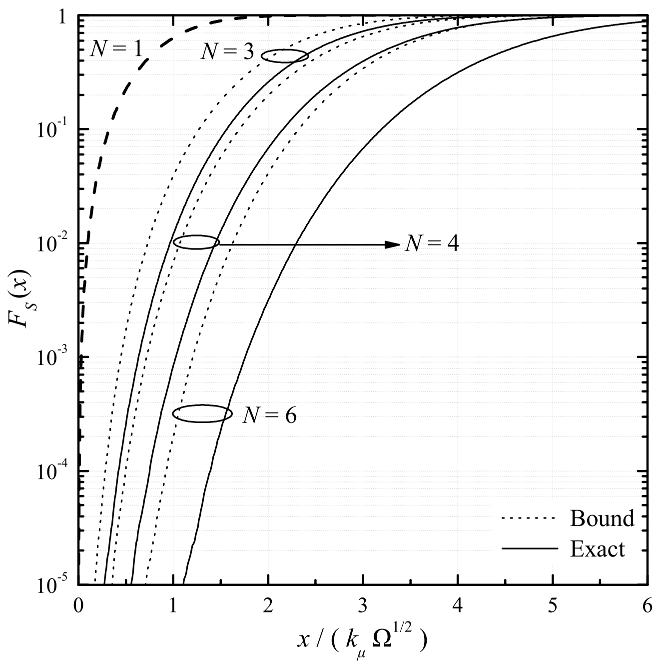


Fig. 2. Bounded and exact CDF of the sum of non-equally weighted Rayleigh distributed RVs.

## III. Numerical Results and Discussion

Having numerically evaluated (20), the bounds for the CDF of the sum of, equally and non-equally, weighted Rayleigh RVs are presented in Figs. 1 and 2, respectively, for moderate values of $N$ with practical interest. In order to verify the tightness of the proposed bound, computer simulations have been also performed and corresponding results for the exact CDF $F_S(\cdot)$ have been included in each figure for comparison purposes. In Fig. 1, the CDF is plotted for $N = 3$, 4, and 6 with $k_\ell = 1\ \forall \ell$, as a function of $x/\sqrt{\Omega}$. As it becomes evident, the proposed bound provides good accuracy, since the numerically evaluating results (dot lines) are close to the exact simulation ones (solid lines) for $F_S(\cdot)$. Moreover, the less the value of $N$, the better accuracy is observed. In Fig. 2, the CDF is plotted for the same values of $N$ with Fig. 1, but with non-equally weighted RVs $k_\ell = \exp[-0.35\,(\ell - 1)]$ $\forall \ell$, as a function of $x/(k_\mu \sqrt{\Omega})$ where $k_\mu^2 = \sum_{i=1}^{N} k_i^2/N$. Once again, note the close match between the proposed bound and the exact simulated curves. Comparing Figs. 1 and 2, it can be observed that the bound is slightly improved in case of non-equally weighted RVs. For example, for $N = 6$ and $F_s(\cdot) = 10^{-5}$, the difference between exact and bound, in Figs. 1 and 2, is 0.46 and 0.31, respectively. It is also evident that for non-equal weights, both bounds and exact curves move towards the curve for $N = 1$ (dash line) in which the bound is identical to the exact as (7) reads.

## IV. Conclusions

With the aid of the well-known arithmetic–geometric mean inequality, a closed-form union upper-bound for the CDF of the weighted sum of $N$ independent Rayleigh distributed fading envelopes, was presented. Comparisons with computer simulation results shown that the proposed bound is tight especially for moderate values of $N$ with practical interest and/or non-equally weighted RVs.

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