Error Rate Analysis of Threshold-Based Hybrid Selection/Maximal-Ratio Diversity over Correlated Nakagami-m Fading Channels

Nikos C. Sagias, *Member, IEEE*, and Kostas Peppas, *Member, IEEE*

***Abstract*— An exact performance analysis of triple-branch threshold-based hybrid selection/maximal-ratio combining (T-HS/MRC) receivers over correlated Nakagami-m fading channels is presented. Our analysis is valid for integer-order fading parameters and an arbitrary covariance matrix. Following the moment-generating function-based approach, the error rate performance of T-HS/MRC receivers for various modulation formats is analytically obtained. Various performance evaluation results are also presented and compared to equivalent simulation ones.**

***Index Terms*— Average symbol error probability (ASEP), correlated statistics, diversity, maximal-ratio combining (MRC), Nakagami-m distribution, selection combining (SC).**

I. Introduction

RECEIVE diversity is one of the most effective techniques to improve the performance of a wireless communication system in a fading environment, with well-known schemes being selection combining (SC), maximal-ratio combining (MRC), and hybrid selection/MRC (HS/MRC) [1], [2]. Recently, a more sophisticated scheme known as threshold-based HS/MRC (T-HS/MRC) has been introduced, where first, the ratio of the instantaneous signal-to-noise ratio (SNR) of each diversity branch to that with the highest value is estimated and compared to a fixed predefined threshold. Then, MRC is applied only to those branches ratios being equal to or higher than this predefined threshold.

In the technical literature, there are several papers dealing with performance analysis of T-HS/MRC receivers. In [3]–[6], the performance of T-HS/MRC with independent diversity branches has been analyzed. The average symbol error probability (ASEP) in equicorrelated Nakagami-m fading has been studied by Zhang and Beaulieu in [7], whereas in [8], the same authors have extended their results assuming an arbitrary correlation matrix using the Green's matrix approximation [9].

This letter deals with the error rate performance of triple-branch T-HS/MRC receivers operating over not necessarily identically distributed (id) Nakagami-m fading channels, with integer-order fading parameters and an arbitrary covariance matrix. More specifically, by extracting the moment-generating function (MGF) of the T-HS/MRC output SNR in terms of rapidly convergent infinite series, an exact analytical solution for the ASEP is derived. The analysis can be applied to several linear modulation schemes including M-ary phase shift keying (M-PSK) and quadrature amplitude modulation (M-QAM).

Manuscript received August 28 2007. The associate editor coordinating the review of this letter and approving it for publication was Dr. George K. Karagiannidis.

The authors are with the Laboratory of Mobile Communications, Institute of Informatics and Telecommunications, National Centre for Scientific Research–"Demokritos," Patriarhou Grigoriou and Neapoleos, Agia Paraskevi, 15310, Athens, Greece (e-mail: nsagias@ieee.org, peppas@telecom.ntua.gr).

Digital Object Identifier 10.1109/LCOMM.2007.071432.

II. Channel and System Model

In this section, the Nakagami-m joint probability density function (PDF) is first presented and then used to asses the T-HS/MRC SNR output statistics. The usual assumptions are made that all channels are noise limited (i.e., additive white Gaussian noise), frequency flat and slow fading, and that the receiver has perfect channels state information.

A. Nakagami-m Joint Statistics

Let γ_1, γ_2, γ_3 be the instantaneous Nakagami-m received SNRs per symbol having an arbitrary covariance matrix $\mathbf{\Sigma}$ with elements $\mathbf{\Sigma}_{\ell,\ell'} = \sqrt{\rho_{\ell,\ell'}\,\overline{\gamma}_\ell\,\overline{\gamma}_{\ell'}/(2\,m^3)}$ $\forall \ell \neq \ell'$ $(\ell,\ell' = 1,2,3)$ and $\mathbf{\Sigma}_{\ell,\ell} = \overline{\gamma}_\ell/m$, with $m = 2,3,\ldots$ denoting the integer-order fading parameter, $\overline{\gamma}_\ell$ being the ℓth average input SNR per symbol, and $\rho_{\ell,\ell'}$ denoting the correlation coefficient between γ_ℓ and $\gamma_{\ell'}$. The joint PDF of γ_1, γ_2, γ_3 is given by [10], [11]

$$\begin{aligned} f_{\gamma_1,\gamma_2,\gamma_3}(\gamma_1,\gamma_2,\gamma_3) &= \frac{\exp\left(-\sum_{\ell=1}^{3} a_\ell\,\gamma_\ell/2\right)}{8\,[\det(\mathbf{\Sigma}/2)]^m\,(b_1\,b_2\,b_3)^{m-1}} \\ &\times \frac{1}{m-1}\sum_{k=m-1}^{\infty} k\,(-1)^{k-m+1}\binom{m+k-2}{2m-3} \\ &\times I_k\left(b_1\sqrt{\gamma_1\,\gamma_2}\right) I_k\left(b_2\sqrt{\gamma_2\,\gamma_3}\right) I_k\left(b_3\sqrt{\gamma_1\,\gamma_3}\right) \end{aligned} \quad (1)$$

where $\det(\mathbf{\Sigma}) = \overline{\gamma}_1\,\overline{\gamma}_2\,\overline{\gamma}_3\,T/\left(2\,m^4\right)$ is the determinant of $\mathbf{\Sigma}$, with $T = 2\,m - \rho_{1,2} - \rho_{2,3} - \rho_{1,3} + \sqrt{2\,\rho_{1,2}\,\rho_{2,3}\,\rho_{1,3}/m}$ and $I_k(\cdot)$ denotes the kth-order modified Bessel function of the first kind. Also in (1), $a_1 = 2m\,(2m - \rho_{2,3})/(T\,\overline{\gamma}_1)$, $a_2 = 2m\,(2m-\rho_{1,3})/(T\,\overline{\gamma}_2)$, $a_3 = 2m\,(2m-\rho_{1,2})/(T\,\overline{\gamma}_3)$,

$$b_1 = \frac{-2\,m}{T\sqrt{\overline{\gamma}_1\,\overline{\gamma}_2}}\left(\sqrt{2\,m\,\rho_{1,2}} - \sqrt{\rho_{2,3}\,\rho_{1,3}}\right) \quad (2a)$$

$$b_2 = \frac{-2\,m}{T\sqrt{\overline{\gamma}_2\,\overline{\gamma}_3}}\left(\sqrt{2\,m\,\rho_{2,3}} - \sqrt{\rho_{1,2}\,\rho_{1,3}}\right) \quad (2b)$$

and

$$b_3 = \frac{-2\,m}{T\sqrt{\overline{\gamma}_1\,\overline{\gamma}_3}}\left(\sqrt{2\,m\,\rho_{1,3}} - \sqrt{\rho_{1,2}\,\rho_{2,3}}\right). \quad (2c)$$

B. T-HS/MRC SNR Output Statistics

Let us define a fixed threshold μ $(0 \le \mu \le 1)$ and also sort γ_1, γ_2, γ_3 in a descending order, e.g., $\gamma_{(1)} \ge \gamma_{(2)} \ge \gamma_{(3)}$, having a joint PDF that can be expressed using (1) as

$$\begin{aligned} &f_{\gamma_{(1)},\gamma_{(2)},\gamma_{(3)}}(x_1,x_2,x_3) \\ &\quad= \sum_{e_\ell \in \mathcal{S}_3} f_{\gamma_1,\gamma_2,\gamma_3}\left(x_{e_\ell[1]}, x_{e_\ell[2]}, x_{e_\ell[3]}\right) \end{aligned} \quad (3)$$

$$\mathcal{M}_{\zeta_L}(s) = \frac{(b_1\, b_2\, b_3)^{1-m}}{8(m-1)\, [\det(\mathbf{\Sigma}/2)]^m} \sum_{k=m-1}^{\infty} \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k(-1)^{k-m+1} \binom{m+k-2}{2m-3} \frac{(b_1/2)^{2\,l+k}\, (b_2/2)^{2\,i+k}\, (b_3/2)^{2\,j+k}}{l!\, i!\, j!\, (l+k)!\, (i+k)!\, (j+k)!} \times [G_L(s;\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) + G_L(s;\, n_1, a_1;\, n_3, a_3;\, n_2, a_2) + G_L(s;\, n_2, a_2;\, n_1, a_1;\, n_3, a_3) + G_L(s;\, n_2, a_2;\, n_3, a_3;\, n_1, a_1) + G_L(s;\, n_3, a_3;\, n_2, a_2;\, n_1, a_1) + G_L(s;\, n_3, a_3;\, n_1, a_1;\, n_2, a_2)] \quad (6)$$

with arguments $x_1 \geq x_2 \geq x_3$, $e_\ell \in \mathcal{S}_3$ denoting $e_\ell = \{e_\ell[1], e_\ell[2], e_\ell[3]\}$, one specific permutation of integers $\{1,2,3\}$. The T-HS/MRC allows the number of combined branches to be a random variable, the value of which is determined according to the following rule: In each sampling period by comparing μ to $\gamma_{(\ell)}/\gamma_{(1)}$, only these branches where $\gamma_{(\ell)} \geq \mu\,\gamma_{(1)}$ holds are coherently combined according to the MRC scheme. More specifically, the instantaneous T-HS/MRC output SNR per symbol is

$$\zeta_L = \sum_{\ell=1}^{L} \gamma_{(\ell)} \quad (4)$$

where $L = 1,2,3$ is an integer variable that represents the number of branches being combined. Hence for $0 < \mu < 1$, we have the following three disjoint events:

- For $\mu\,\gamma_{(1)} > \gamma_{(2)} \geq \gamma_{(3)}$, $L = 1$,
- For $\gamma_{(1)} > \gamma_{(2)} \geq \mu\gamma_{(1)} > \gamma_{(3)}$, $L = 2$, and
- For $\gamma_{(1)} \geq \gamma_{(2)} \geq \gamma_{(3)} \geq \mu\,\gamma_{(1)}$, $L = 3$.

Also, setting $\mu = 0$ and 1, the MRC and SC schemes may be employed, respectively.

III. ASEP of Triple-Branch T-HS/MRC Receivers

The probability space of the three events presented in the previous section can be formed joining all partitions [7]. Hence, the ASEP of T-HS/MRC can be calculated as

$$\overline{P}_{se} = \sum_{\ell=1}^{3} \Pr(se, L = \ell) \quad (5)$$

where $\Pr(se, L = \ell)$ is the ASEP of T-HS/MRC when the event that ℓ branches that satisfy the threshold conditions are selected, occurs. For each event of L, the probability $\Pr(se, L = \ell)$ can be separately calculated following the MGF-based approach.

A. Error Probability Analysis

Using (3), (4), the definition of MGF, and an infinite series representation for Bessel functions [12, eqs. (8.445), (8.350/2), and (8.352/2)], the MGF of ζ_L for each one of the three possible values of L can be represented by a unified analytical expression which includes only elementary functions and is given by (6) (at top of this page), with $n_1 = l + j + k$, $n_2 = l + i + k$, $n_3 = i + j + k$,

$$G_1(s;\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = n_3!\, 2^{2+n_2+n_3}\, a_3^{-n_3-1} \times \left[\frac{n_1!\, n_2!\, a_2^{-n_2-1}}{(s + a_1/2)^{n_1+1}} - \frac{n_2!}{a_2^{n_2+1}} \sum_{p_1=0}^{n_2} \left(\mu\, \frac{a_2}{2}\right)^{p_1} \frac{(p_1+1)_{n_1}}{(s+B_1)^{u_1}} - \frac{n_1!}{(s + a_1/2)^{n_1+1}} \sum_{p_1=0}^{n_3} \frac{a_3{}^{p_1}\, (p_1+1)_{n_2}}{(a_2+a_3)^{n_2+p_1+1}} + \sum_{p_1=0}^{n_3} \sum_{p_2=0}^{n_2+p_1} \frac{(\mu/2)^{p_2}\, (p_1+1)_{n_2}\, (p_2+1)_{n_1}}{a_3{}^{-p_1}\, (a_2+a_3)^{u_6}\, (s+B_2)^{u_2}} \right] \quad (7a)$$

$$G_2(s;\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = n_2!\, n_3!\, (a_3/2)^{-n_3-1} \times \left\{ \sum_{p_2=0}^{n_2} \frac{(p_2+1)_{n_1}}{(s + a_2/2)^{u_3}} \left[\frac{\mu^{p_2}\, (\mu+1)^{-u_2}}{[s + (a_1 + \mu a_2)/(2\mu+2)]^{u_2}} - \frac{2^{-u_2}}{[s + (a_1+a_2)/4]^{u_2}} \right] - \sum_{p_1=0}^{n_3} \sum_{p_2=0}^{n_2} \frac{(\mu\, a_3/2)^{p_1}}{p_1! p_2!} \times \frac{(n_1+p_1+p_2)!}{(s+a_2/2)^{u_3}} \left[\frac{\mu^{p_2}\, (\mu+1)^{-u_4}}{(s+B_3)^{u_4}} - \frac{2^{-u_4}}{(s+B_4)^{u_4}} \right] \right\} \quad (7b)$$

and

$$G_3(s;\, n_1, a_1;\, n_2, a_2;\, n_3, a_3) = n_2!\, n_3! \sum_{p_1=0}^{n_3} \sum_{p_2=0}^{n_2} \frac{\mu^{p_1}}{p_1!\, p_2!} \times (p_1+p_2+n_1)!\, \frac{(s + a_3/2)^{-u_5}}{(s+a_2/2)^{u_3}} \left[\frac{(2\mu+1)^{-u_4}}{(s+B_5)^{u_4}\, \mu^{-p_2}} - \frac{(\mu+2)^{-u_4}}{(s+B_6)^{u_4}} \right] - n_3! \sum_{p_1=0}^{n_3} \sum_{p_2=0}^{n_2+p_1} \frac{(p_1+1)_{n_2}\, (p_2+1)_{n_1}}{[s + (a_2+a_3)/4]^{u_6}} \times \frac{2^{-u_6}}{(s+a_3/2)^{u_5}} \left[\frac{\mu^{p_2} (2\mu+1)^{-u_2}}{(s+B_5)^{u_2}} - \frac{3^{-u_2}}{(s+B_7)^{u_2}} \right] \quad (7c)$$

with $(i)_j = (i+j-1)!/(i-1)!$ standing for the Pochhammer symbol (i, j positive integers). Also in (7), $u_1 = n_1 + p_1 + 1$, $u_2 = n_1 + p_2 + 1$, $u_3 = -p_2 + n_2 + 1$, $u_4 = n_1 + p_1 + p_2 + 1$, $u_5 = -p_1 + n_3 + 1$, $u_6 = -p_2 + p_1 + n_2 + 1$,

$$B_j = \frac{a_1}{2} + \frac{\mu}{2} \sum_{i=2}^{j+1} a_i,\ \ (j = 1,2),\ \ B_3 = \frac{\mu a_2 + \mu a_3 + a_1}{2\mu + 2}, \quad (8a)$$

$$B_4 = \frac{a_1 + a_2 + \mu a_3}{4},\ \ B_5 = \frac{\mu a_2 + \mu a_3 + a_1}{4\mu + 2}, \quad (8b)$$

$$B_6 = \frac{a_2 + \mu a_3 + a_1}{2\mu + 4},\ \text{and}\ B_7 = \sum_{i=1}^{3} \frac{a_i}{6}. \quad (8c)$$

TABLE I
NUMBER OF REQUIRED TERMS FOR CONVERGENCE OF THE ABEP OF BDPSK BASED ON (5) AND (6) WITH $e_r \leq 5\%$.

$\overline{\gamma}_b$ (dB)	$m=2$		$m=3$		$m=4$	
μ :	0.5	0.8	0.5	0.8	0.5	0.8
-5	3	3	4	4	4	6
0	3	3	4	3	4	5
5	2	3	3	3	4	4
10	1	2	3	3	4	4
15	1	1	3	3	3	3

Note that for $\mu = 0$, (6) numerically agrees with $\mathcal{M}_{\gamma_{\mathrm{mrc}}}(s) = [\det(\mathbf{I}_3 + s\,\mathbf{\Sigma})]^{-m}$ for MRC receivers, with $\mathbf{I}_3$ being the 3×3 identity matrix.

Based on (5) and (6), the average bit error probability (ABEP) for non-coherent binary frequency shift keying (NBFSK) and binary differential phase shift keying (BDPSK) modulation signallings can be directly calculated (e.g. for BDPSK, $\overline{P}_{be} = 0.5 \sum_{L=1}^{3} \mathcal{M}_{\zeta_L}(1)$). For other types of modulation formats, such as M-PSK and M-QAM, single integrals with finite limits and integrands composed of elementary (exponential and trigonometric) functions have to be readily evaluated via numerical integration.

B. Numerical and Computer Simulation Results

Numerical and computer simulation results for T-HS/MRC receivers operating over linearly correlated ($\rho_{1,2} = \rho_{2,3} = 0.795$ and $\rho_{1,3} = 0.605$) and non id Nakagami-m fading channels are provided, assuming an exponential power decaying profile $\overline{\gamma}_\ell = \overline{\gamma}_1 \exp[-0.1\,(\ell - 1)]$.

Setting equal summation limits for the truncation of (5), (6) to all sums, Table I summarizes the number of terms needed so as the ABEP of BDPSK to converge with relative error $e_r \leq 5\%$ comparing with accurate computer simulations. An increase on the first branch average SNR per bit, $\overline{\gamma}_b = \overline{\gamma}_1 / \log_2(M)$, results to a decrease of the required number of terms, while for a fixed $\overline{\gamma}_b$, the required number of terms for convergence slightly increases with increasing m and/or μ. Also, from additional convergence experiments that were conducted it was observed that an increase on the value of any of the correlation coefficients results to a small increase of the required summation terms.

In Fig. 1, a few curves for the ABEP, $\overline{P}_{be} = \overline{P}_{se} / \log_2(M)$, for Gray-encoded square M-QAM modulation format are plotted as a function of $\overline{\gamma}_b$ for $m = 2$ and various μ and M. As expected, the ABEP improves as μ, M decrease and/or $\overline{\gamma}_b$ increases. In the same figure, the numerically evaluated results are compared to equivalent simulation ones. These comparisons clearly show that all curves for the ABEP coincide with square pattern signs obtained via simulations, verifying the correctness of the presented analysis.

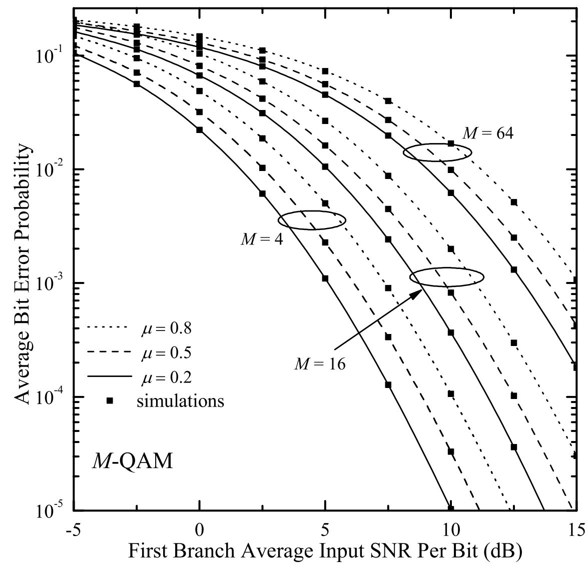

Fig. 1. ABEP of triple-branch T-HS/MRC for Gray-encoded square M-QAM modulation with $m = 2$ and a linearly arbitrary correlation model as a function of the first branch average input SNR per bit.

IV. CONCLUSIONS

By following the MGF-based approach, infinite series representations for the error rate performance of the triple-branch T-HS/MRC receivers over arbitrarily correlated Nakagami-m fading channels were obtained. Extensive numerical and computer simulation results were presented and compared, and not only a perfect match was observed, but also the infinite series were shown to be rapidly convergent.

REFERENCES

[1] R. K. Mallik and M. Z. Win, "Analysis of hybrid selection/maximal-ratio combining in correlated Nakagami fading," *IEEE Trans. Commun.*, vol. 50, no. 8, pp. 1372–1383, Aug. 2002.

[2] R. K. Mallik, P. Gupta, and Q. T. Zhang, "Minimum selection GSC in independent Rayleigh fading," *IEEE Trans. Veh. Technol.*, vol. 54, no. 3, pp. 1013–1021, May 2005.

[3] M. K. Simon and M.-S. Alouini, "Performance analysis of generalized selection combining with threshold test per branch (T-GSC)," *IEEE Trans. Veh. Technol.*, vol. 51, no. 5, pp. 1018–1029, Sept. 2002.

[4] A. Annamalai, G. Deora, and C. Tellambura, "Unified analysis of generalized selection diversity with normalized threshold test per branch," in *Proc. IEEE Wirel. Commun. Netw. Conf.*, vol. 1, Mar. 2003, pp. 752–756.

[5] L. Xiao and X. Dong, "Unified analysis of generalized selection combining with normalized threshold test per branch," *IEEE Trans. Wireless Commun.*, vol. 5, no. 8, pp. 2153–2163, Aug. 2006.

[6] X. D. Zhang and N. C. Beaulieu, "SER and outage of threshold based hybrid selection/maximal ratio combining over generalized fading channels," *IEEE Trans. Commun.*, vol. 52, no. 12, pp. 2143–2153, Dec. 2004.

[7] ——, "SER of threshold based hybrid selection/maximal ratio combining in equicorrelated Nakagami fading," *IEEE Commun. Lett.*, vol. 8, no. 9, pp. 552–554, Sept. 2004.

[8] ——, "SER of threshold based hybrid selection/maximal ratio combining in correlated Nakagami fading," *IEEE Trans. Commun.*, vol. 53, no. 9, pp. 1423–1426, Sept. 2005.

[9] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "An effiicient approach to multivariate Nakagami-m distribution using Green's matrix approximation," *IEEE Trans. Wireless Commun.*, vol. 2, no. 5, pp. 883–889, Sept. 2003.

[10] M. Hagedorn, P. J. Smith, P. J. Bones, R. P. Millane, and D. Pairman, "A trivariate chi-square distribution derived from the complex Wishart distribution," *J. Multivariate Anal.*, vol. 97, no. 3, pp. 655–674, Mar. 2006.

[11] K. D. P. Dharmawansa, R. M. A. P. Rajatheva, and C. Tellambura, "Infinite series representations of the trivariate and quadrivariate Nakagami-m distributions," in *Proc. IEEE Inter. Conf. Commun.*, June 2007, pp. 1114–1118.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 6th ed. New York: Academic Press, 2000.